Standard X

MATHEMATICS

Part-2

Government of Kerala
Department of Education

State Council of Educational Research and Training (SCERT)
2016

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear children,

Mathematics starts in counting and measuring. In the age of agriculture, it becomes the second degree equations of areas; rises to astronomy for weather prediction. Grows into the branch of mathematics called trigonometry. In Renaissance Europe, trigonometry forms the foundation of navigation. It becomes the basis of locating places using satellites in today's world. The mathematical principles which seventeenth century mathematicians developed as purely mathematical operations of numbers are now used to make security systems in e-transactions. I wish all of you would recognize the innumerable applications of mathematics and revel in its theoretical rhythms.

With love and regards

Dr. P. A. Fathima
Director, SCERT

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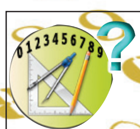
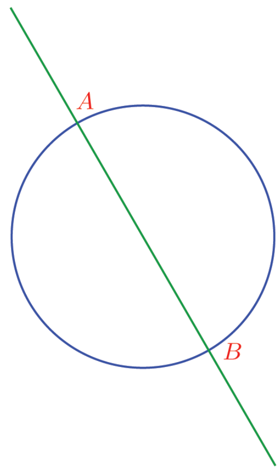


For Discussion

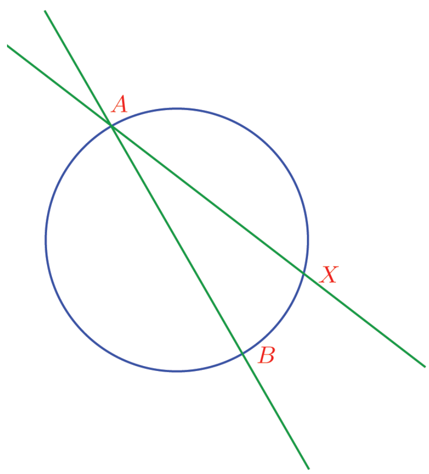
Line and circle

See this picture :

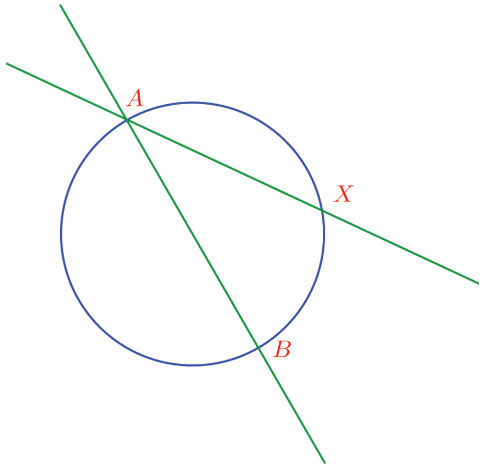


AB is the diameter through the point *A* on the circle; and it is extended a bit on either side.

This picture shows another chord through *A*, instead of a diameter, which is also extended.

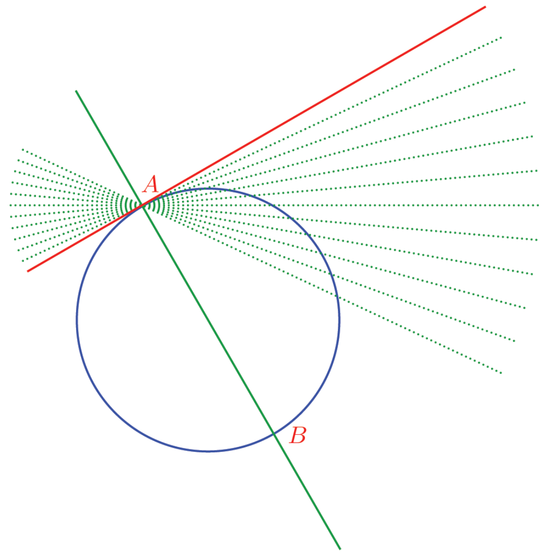


Without altering the position of A, suppose we make X closer to A.



What if we make X closer and closer to A along the circle?

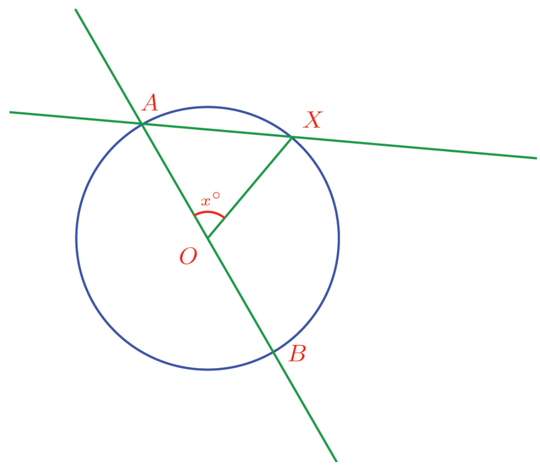
Draw a circle centered at a point O in GeoGebra and mark points A, X on it. Draw lines joining O, A and A, X. What happens to the line AX, when X is moved closer to A? When X coincides with A? Join OX. What happens to the angles OAX and OXA as X is moved closer and closer to A?



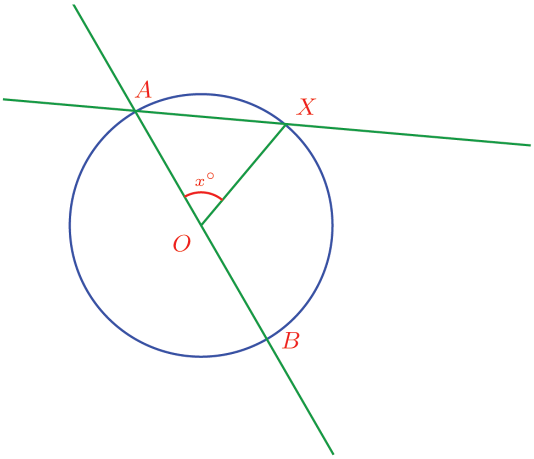
The red line in the picture just touches the circle at A, right?

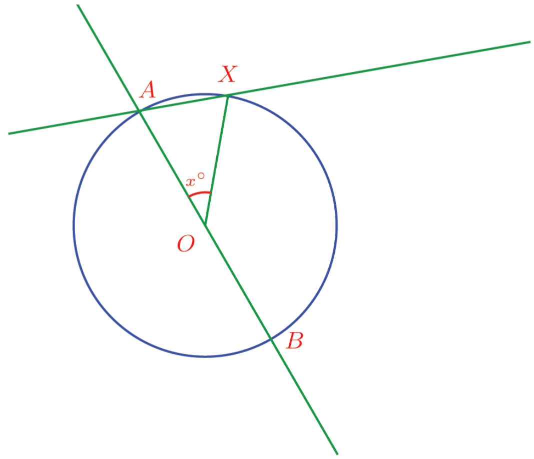
This line is called the *tangent* to the circle at A. Look at the picture again; see any relation between the tangent and the diameter?

To make this clear, let's take the central angle of chord AX as x°.



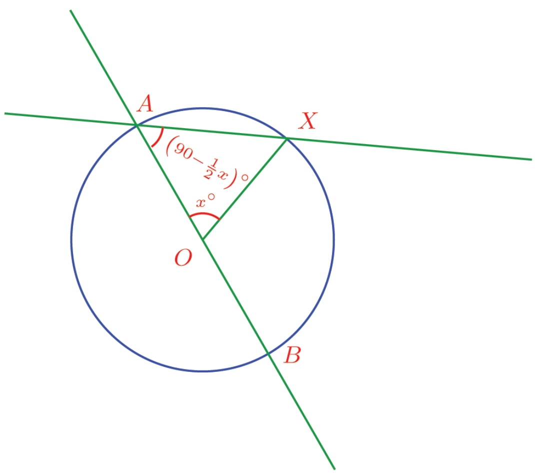
As X gets closer to A, the length of the chord AX and its central angle becomes smaller; that is, the number x gets closer to zero.





What about the angle between the chord and the diameter? Since ΔAOX is isosceles, this angle is

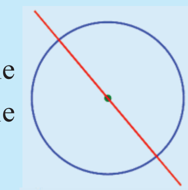
$$\frac{1}{2}(180 - x)^{\circ} = \left(90 - \frac{1}{2}x\right)^{\circ}$$



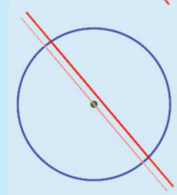
As X gets closer to A, this angle gets closer to 90°. And when the extended chord becomes a tangent, the angle becomes exactly 90°.

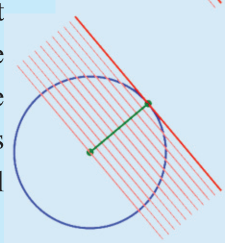
Sliding line

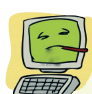
See this picture – a circle and a line through the centre.



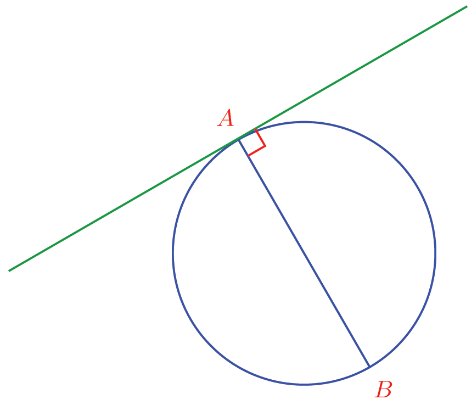
Suppose we slide the line a little higher. And sliding it more and more, we finally get a line just touching the circle at a single point. And the line joining the centre and this point is perpendicular to all these parallel lines.







Draw a circle and a radius in GeoGebra. Choose a point on the radius and draw the perpendicular through it. Now change the position of the point. What happens when it is on the circle?



We state this as a general principle:

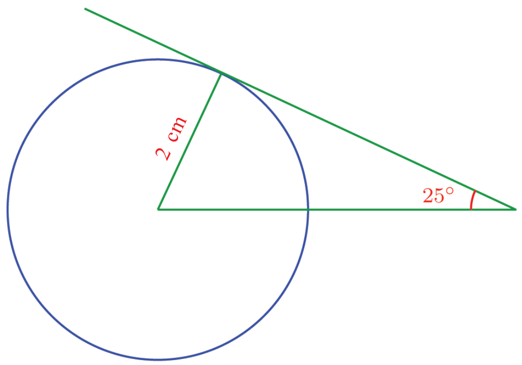
The tangent at a point on a circle is perpendicular to the diameter through that point.

To draw a tangent to a circle in GeoGebra, choose **Tangents.** Click on the circle and a point on the circle. What if we click on a point outside the circle?

Draw the tangent at a point on a circle and enable **Trace On** for it and **Animation** for the point. What do we get?

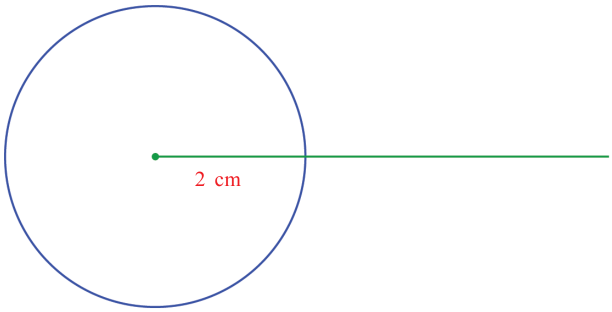
Let's look at some problems based on this.

In the picture below, the top line is a tangent to the circle:



Can you draw this picture in your notebook?

Draw a circle of radius 2 centimetres and a horizontal line through its centre.

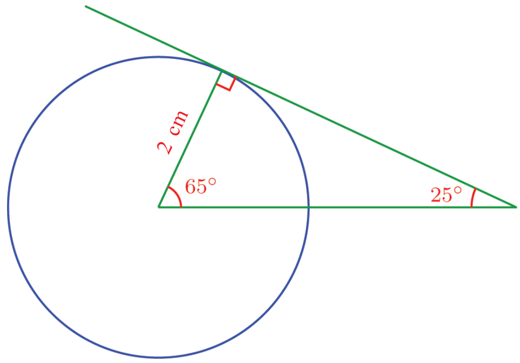


Now at what point on the circle do we draw the required tangent?

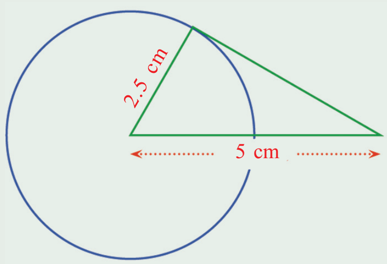
Look at the first picture again; the top angle of the triangle is 90°; and another angle is 25°.

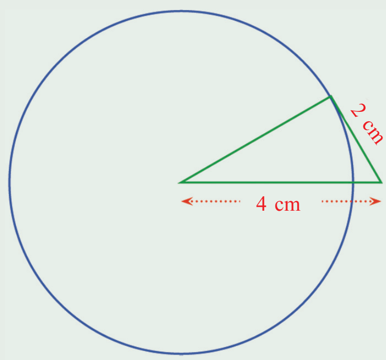
So, the third angle is 65°.

Can't you complete the picture now?



(1) In each of the two pictures below, a triangle is formed by a tangent to a circle, the radius though the point of contact and a line through the centre:

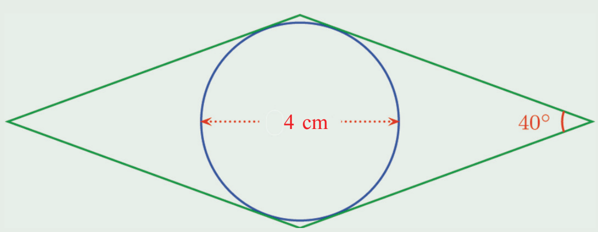




Draw these in your note book.

(2) In the picture, all sides of a rhombus are tangents to a circle.

Draw this picture in your notebook.

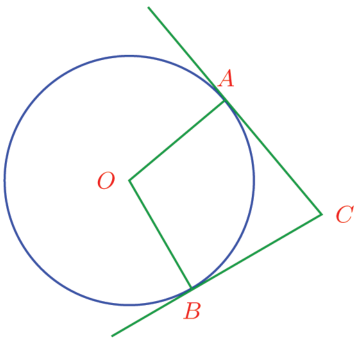


(3) What sort of a quadrilateral is formed by the tangents at the ends of two diameters of a circle?

(4) Prove that the tangents drawn to a circle at the two ends of a diameter are parallel.

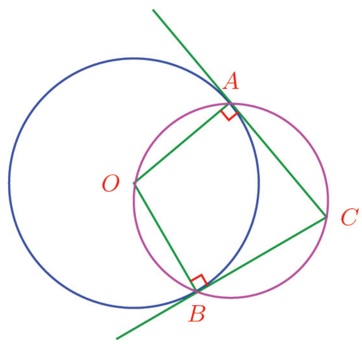
Tangents and angles

See this picture:



The tangents at the points *A*, *B* on a circle centered at *O* meet at *C*.

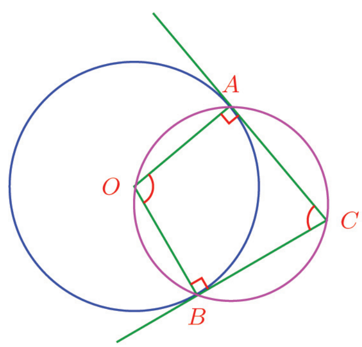
In the quadrilateral *OACB*, the angles at the opposite corners *A*, *B* are right; so their sum is 180°. Thus the quadrilateral is cyclic.



That is,

The quadrilateral with vertices at the centre of a circle, two points on it and the point where the tangents at these points meet, is cyclic.

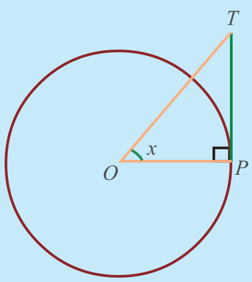
In such a quadrilateral the sum of the other two angles is also 180°.



The name

The name tangent is derived from the Latin word *tangere*, meaning 'to touch'.

The full name of the tan measure is also tangent. What is its connection with a line touching a circle?

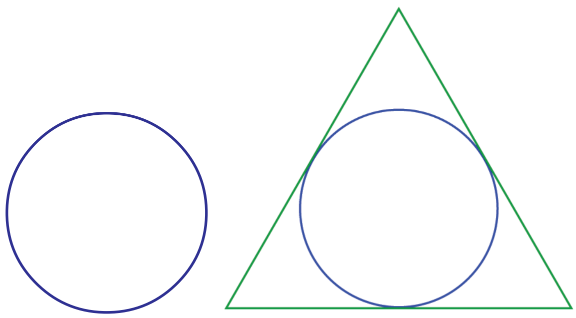


If we take the radius of the circle as the unit of length, then the length of the tangent *PT* is tan *x*, right?

This also is a general idea worth noting:

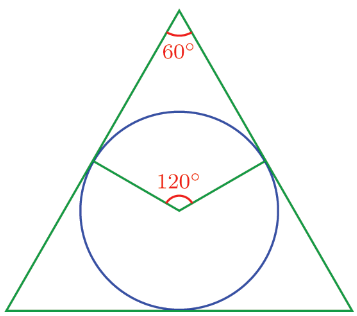
In a circle, the angle between the radii through two points and the angle between the tangents at these points are supplementary.

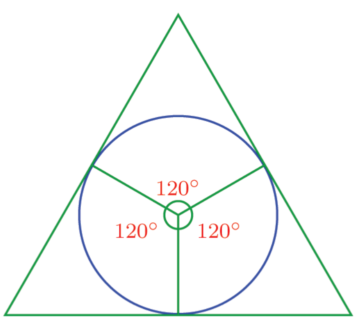
Let's look at some figures which can be drawn using these ideas. First we draw an equilateral triangle, exactly covering a circle.



The sides of the triangle must be tangents to the circle. And since the triangle is to be equilateral, the angle between them must be 60°.

What about the angle between the radii through the points of contact of the tangents?

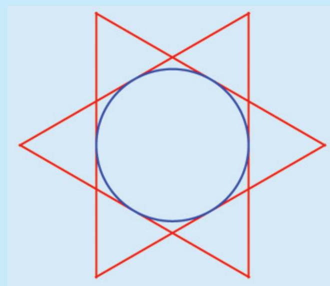




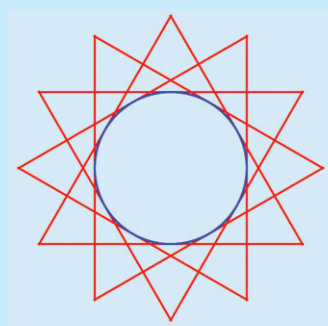
Thus we can see that all the angles between the radii through the points of contact are 120°.

Circle with lines

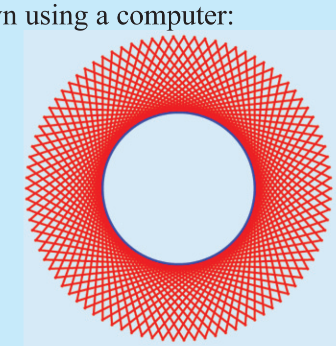
The picture below shows a star made by six tangents to a circle.



How about 12 tangents?

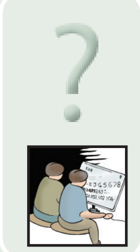


The picture below shows 90 tangents drawn using a computer:



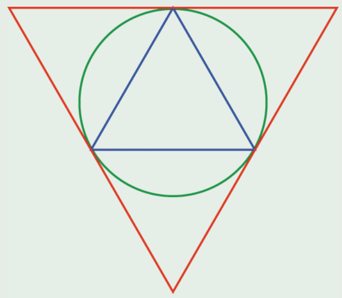
Thus we need only draw three radii of the circle 120° apart and draw the tangents at their ends to get our triangle.

Draw a circle of radius 3 centimetres and draw an equilateral triangle like this.



(1) Draw a circle of radius 2.5 centimetres. Draw a triangle of angles 40°, 60°, 80° with all its sides touching the circle.

(2) In the picture, the small (blue) triangle is equilateral. The sides of the large (red) triangle are tangents to the circumcircle of the small triangle at its vertices.

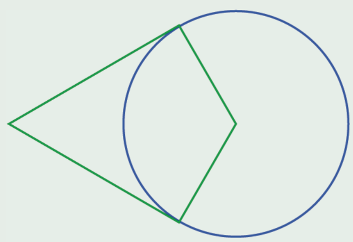


i) Prove that the large triangle is also equilateral and its sides are double that of the small triangle.

ii) Draw this picture, with sides of the smaller triangle 3 centimetres.

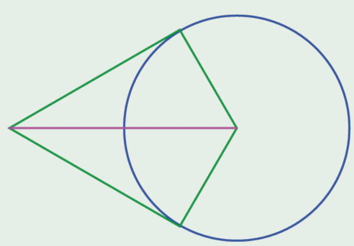
iii) Instead of an equilateral triangle, if we draw the tangents to the circumcircle of any other triangle at its vertices, do we get a similar triangle with double the sides? Justify.

(3) The picture shows the tangents at two points on a circle and the radii through the points of contact.

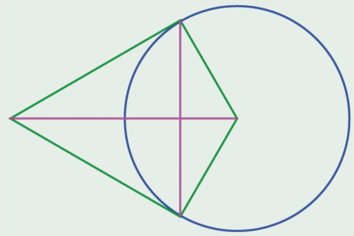


i) Prove that the tangents have the same length.

ii) Prove that the line joining the centre and the points where the tangents meet bisects the angle between the radii.

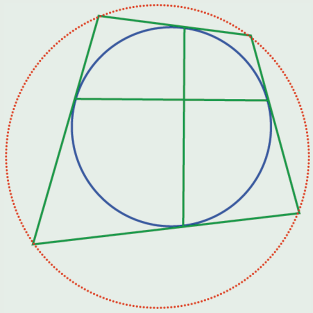


iii) Prove that this line is the perpendicular bisector of the chords joining the points of contact.



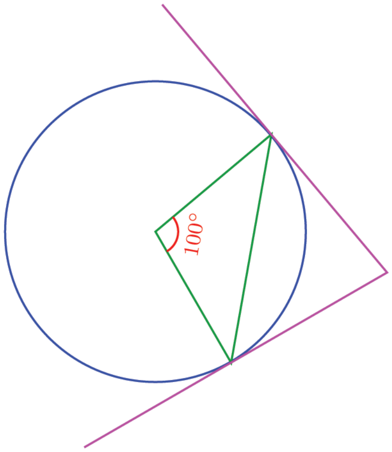
(4) Prove that the quadrilateral with sides as the tangents at the ends of a pair of perpendicular chords of a circle is cyclic.

What sort of a quadrilateral do we get if one chord is a diameter? And if both chords are diameters?

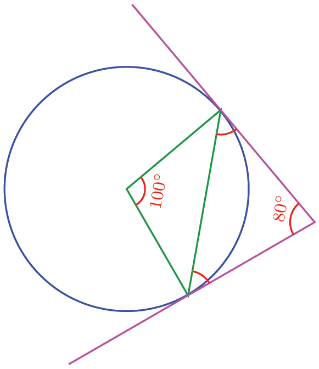


Chord and tangent

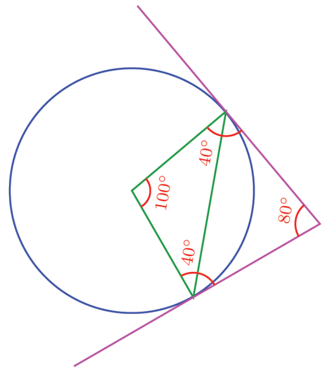
The picture shows the tangents at the two ends of a chord of a circle:



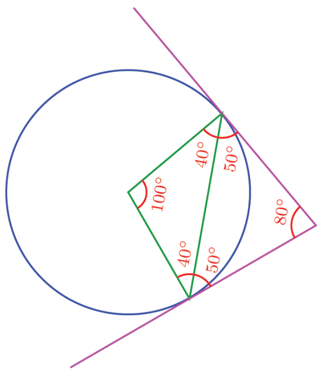
We know that angle between the tangents is 80°. What about the angles between the tangents and the chord?



The two sides of the green triangle in the picture are equal and so are the angles opposite them. Since the sum of these angles is $180° - 100° = 80°$, each measure $40°$.



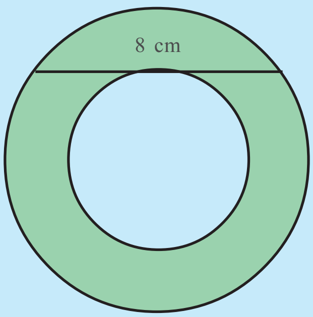
The angle between the radius and the tangent is 90°. So the angle between the chord and the tangent is $90° - 40° = 50°$.



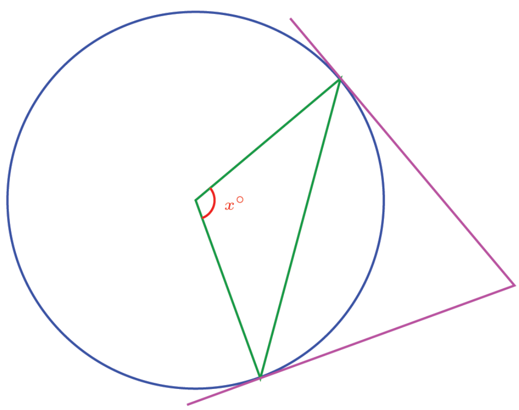
This is half the central angle of the chord, isn't it?

Area problem

What is the area of the green ring?

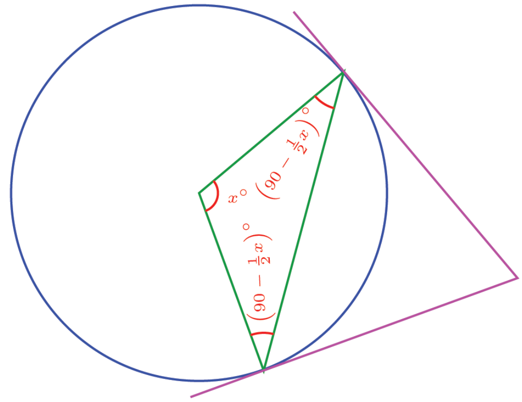


Is it true for any chord? To check this, let's take the central angle as $x°$.

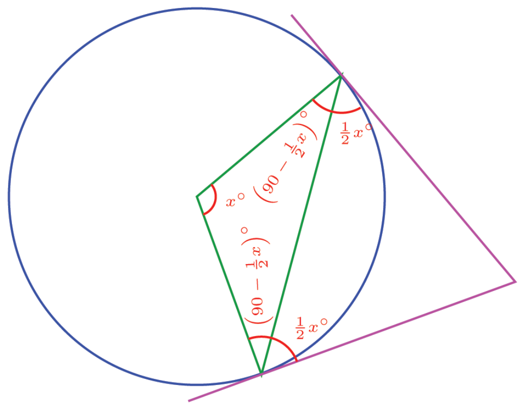


So the other two angles of the green triangle are

$$\frac{1}{2}(180 - x)^\circ = \left(90 - \frac{1}{2}x\right)^\circ$$



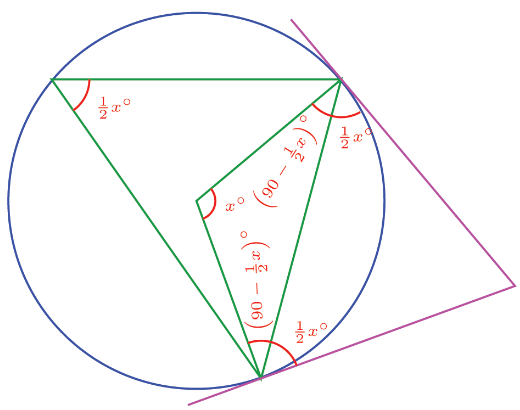
Since the angle between the tangent and the radius is 90°, we can see that the angle between the tangent and chord is $\frac{1}{2}x^\circ$.



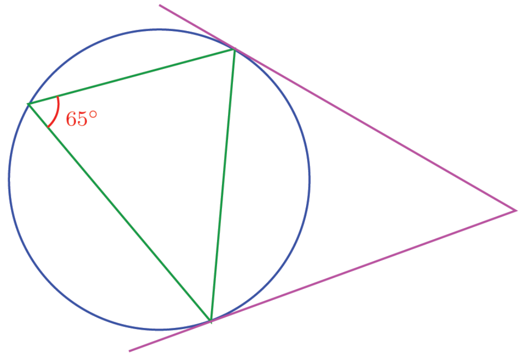
In GeoGebra, draw a circle and a chord. Draw the tangents at the ends of this chord. Mark the central angle of the chord and the angles between the chord and the tangents. What is the relation between these angles? Draw several chords and see.

In a circle, the angle between a chord and tangent at either end is half the central angle of the chord.

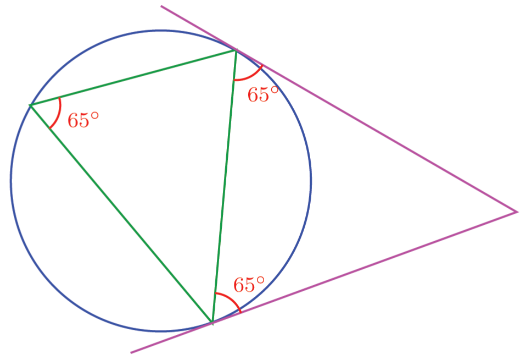
The angle made by the chord on the larger part of the circle is also half the central angle, isn't it?



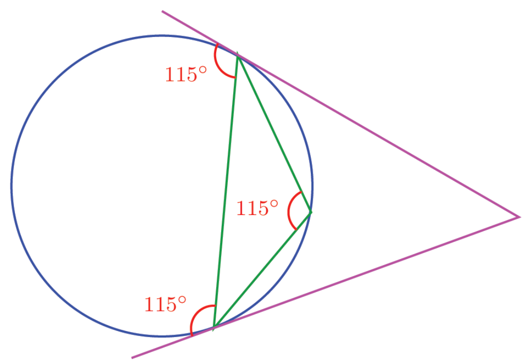
So in this picture, what is the angle which the tangents make with the chord?



The angles on the right are 65°.



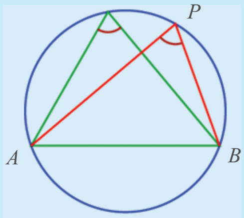
The angles which the tangents make on the left of the chord are 180° − 65° = 115°. This is the angle which the chord makes on the smaller part of the circle, isn't it?



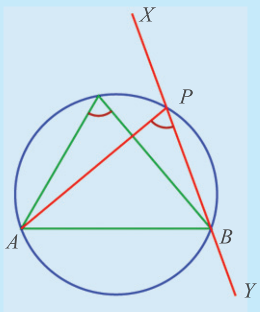
So the relation between the angles which the chord makes with the tangents at its ends and the angles which it makes on the circle can be shown like this:

Unchanging angle

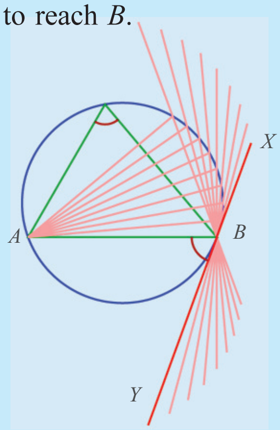
We have seen that the angles on the same part of a circle are equal.



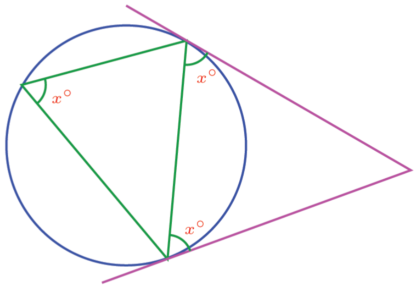
Let's extend *PB* a bit.

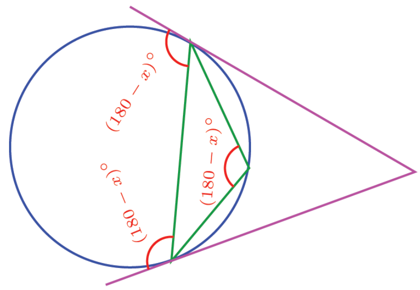


Now suppose *P* moves along the circle to reach *B*.



The line *XY* becomes the tangent at *B*. And the angle does not change.

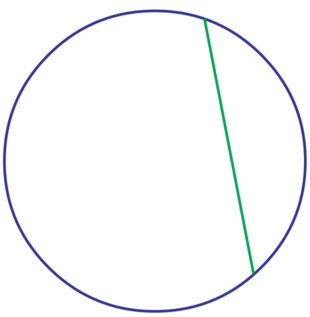


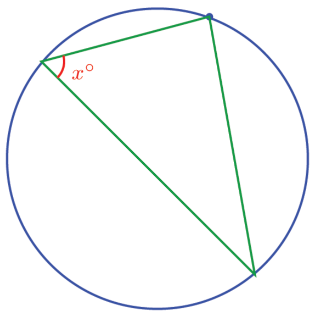


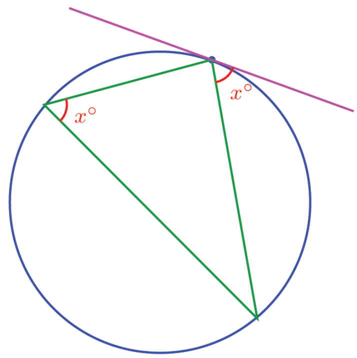
And we can write it like this:

In a circle, the angles which a chord makes with the tangents at its ends on any side are equal to the angle which it makes on the part of the circle on the other side.

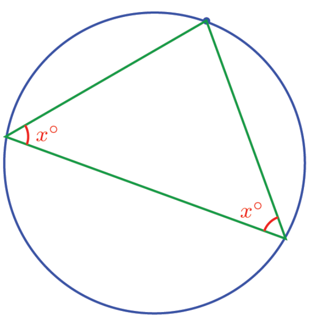
We draw a tangent to a circle at a point, by drawing the perpendicular to the diameter through this point, right? The above idea can be used to draw tangents, even if the centre is not known. We need only draw a chord through this point and draw the angle which it makes on one part of a circle on the opposite side of the chord:

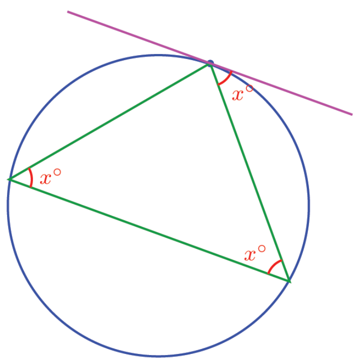






If the triangle drawn is isosceles, then we need draw only the line through the point, parallel to the bottom side.

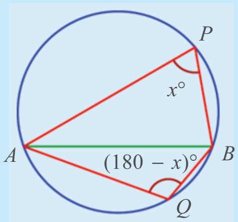




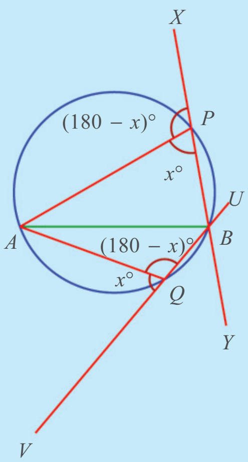
So, to draw the tangent to a circle at a specified point, we first draw an arc of a circle centred at this point and join the points where it cuts the first circle:

Flip – flop

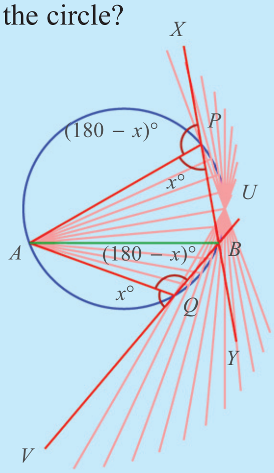
Angles on the two parts of a circle are supplementary:



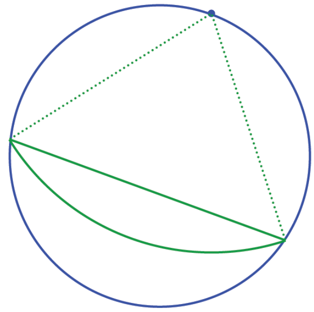
Let's extend the lines as before.



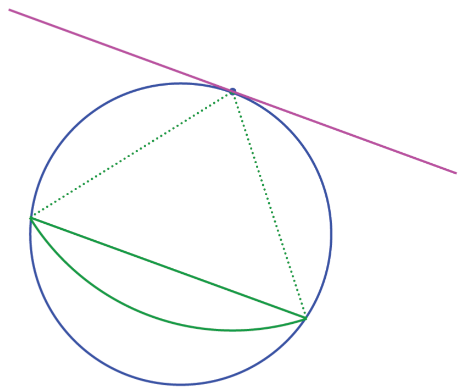
What happens as P moves towards along the circle?

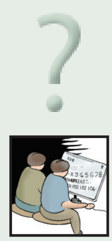


Throughout the motion, $x°$ is below AP and $(180 - x)°$ above AP, isn't it?

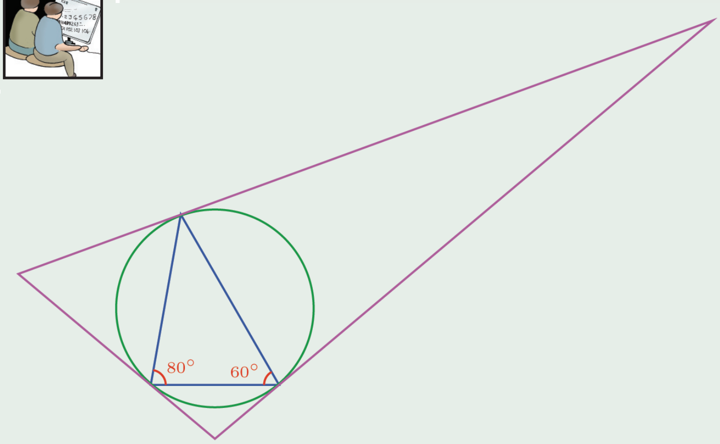


Now we need only draw a line parallel to this line through the point at which we want the tangent:





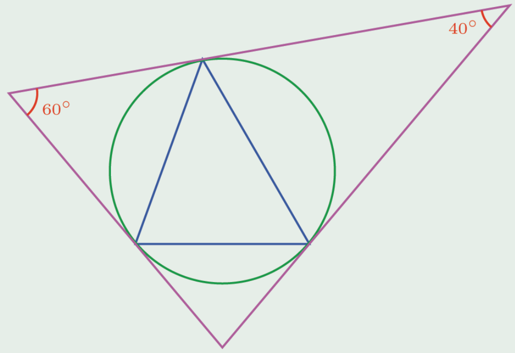
(1) In the picture, the sides of the large triangle are tangents to the circumcircle of the small triangle, through its vertices.



Calculate the angles of the large triangle.

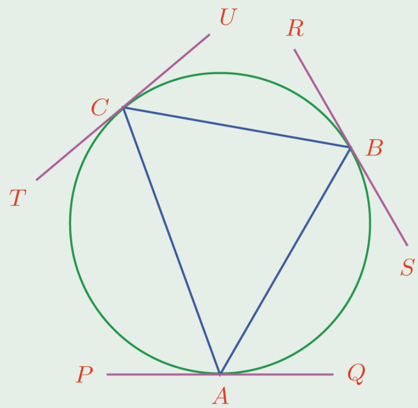
(2) In the picture, the sides of the large triangle are tangents of the circumcircle of the smaller triangle, through its vertices.

Calculate the angles of the smaller triangle.



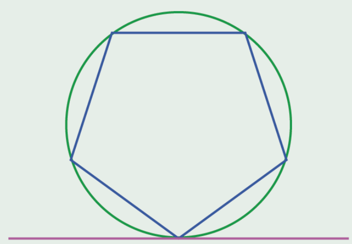
(3) In the picture, PQ, RS, TU are tangents to the circumcircle of ΔABC.

Sort out the equal angles in the picture.



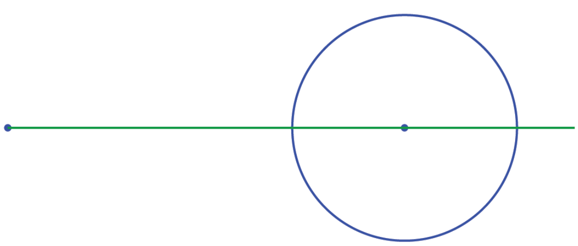
(4) In the picture, the tangent to the circumcircle of a regular pentagon through a vertex is shown.

Calculate the angle when the tangent makes with the two sides of the pentagon through the point of contact.



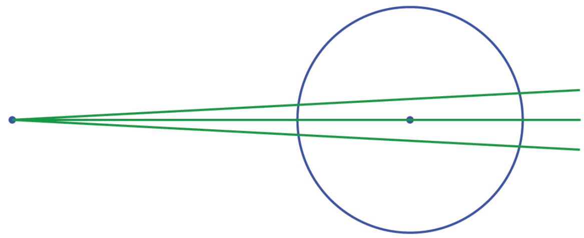
A tangent from outside

See this picture:

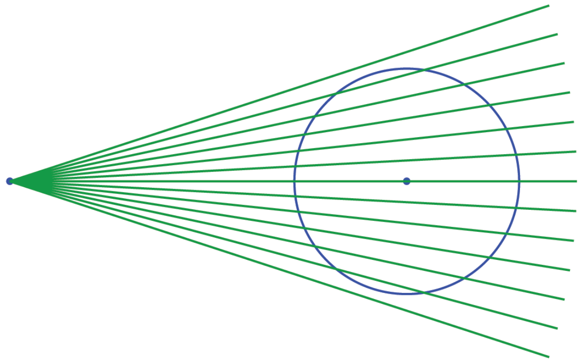


A point outside a circle is joined to the centre and extended. It cuts the circle at two points; and these points are the ends of a diameter.

Suppose we join the same point outside the circle to a point a little above or below the centre?

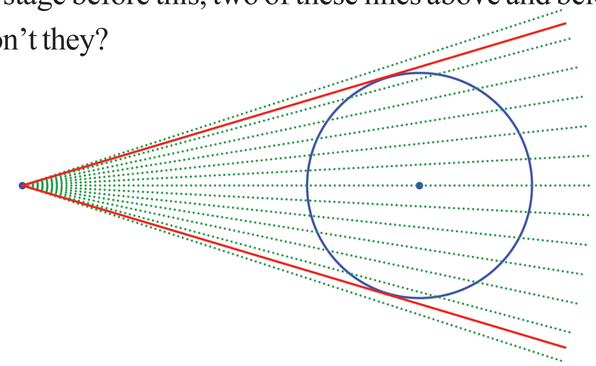


The points where the line cuts the circle gets a little closer. Let's continue this:



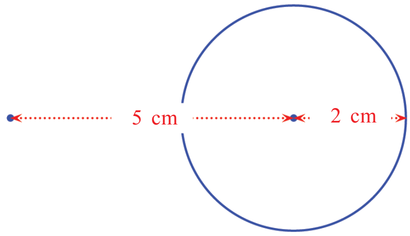
The lines which cut the circle at closer and closer points, leave the circle entirely after a stage.

But at some stage before this, two of these lines above and below just touch the circle, don't they?



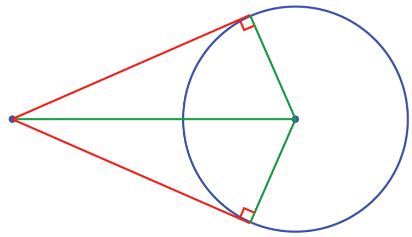
From a point outside a circle, two tangents can be drawn.

But we haven't discussed say how we can actually draw such a pair of tangents. See this picture:



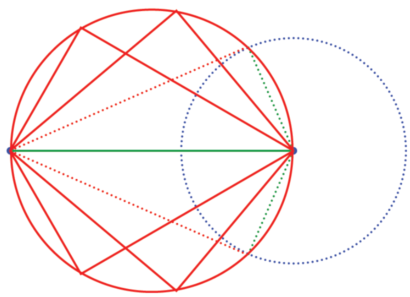
A point is marked 5 centimetres away from the centre of a circle of radius 2 centimetres

How do we draw the pair of tangents to the circle from this point?



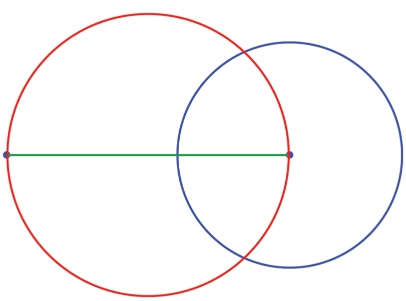
Perhaps it would be clear if we imagine how the picture would be after they are drawn;

We need two pairs of mutually perpendicular lines from the centre of the circle and the point outside.



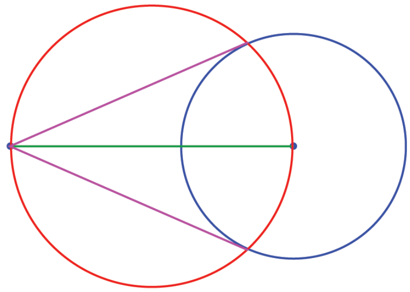
We have seen in the lesson **Circles**, that all such pairs of mutually perpendicular lines meet on the circle with the line joining these points as diameter:

In those pairs we want, one line should be a radius of our original circle; that is, the lines should meet on this circle. For that, we

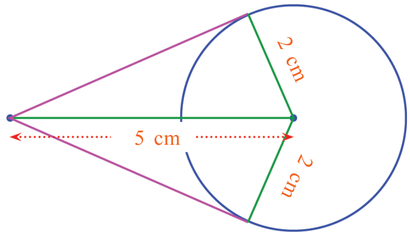


Draw a circle centred at a point O in GeoGebra and mark points A, B on it. Draw the tangents at these points and mark their point of intersection as C. Draw the quadrilateral OACB. Is it cyclic? We can check by drawing the circle through O, A, B using. **Circle Through Three Points**. Move A, B and see what happens when they get closer and farther apart. What happens when they are the ends of a diameter?

Joining the points of intersection of these circles to the point outside, we get the tangents from it:



In our problem, the radius of the original circle is 2 centimetres and the distance from the centre to the point outside is 5 centimetres.



So, we can calculate the lengths of the tangents using Pythagoras Theorem:

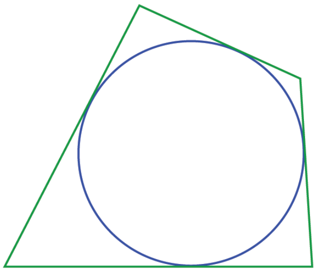
$$\sqrt{5^2 - 2^2} = \sqrt{21} \text{ centimetres}$$

We have already seen that if tangents are drawn from two points on a circle, then their lengths from the point of contact to the point of intersection are equal. We can now state it like this:

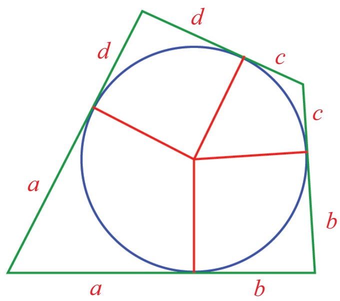
The tangents to a circle from a point are of the same length.

Let's look at a problem based on this. The picture shows the quadrilateral formed by the tangents at four points on a circle.

Let's join the centre to these points.



Taking the lengths of the tangents from the corners as *a*, *b*, *c*, *d*, we can mark these lengths as below:



So, the sum of the lengths of the bottom and top sides of the quadrilateral is $(a + b) + (c + d)$.

What about the sum of the left and right sides?
$(a + d) + (b + c)$

Both sums are $a + b + c + d$. Thus we have the following:

In a quadrilateral formed by the tangents at four points on a circle, the sum of the opposite sides are equal.

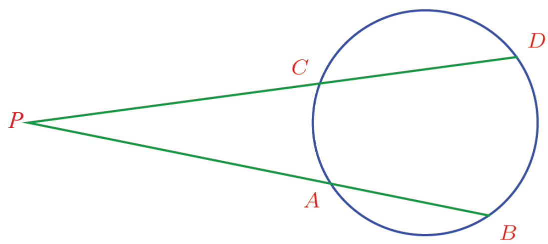
Draw a circle in GeoGebra and mark four points on it. Draw tangents at these points and mark their points of intersection. Draw the quadrilateral with these as vertices. Then we can hide the tangents. Mark the lengths of the sides of the quadrilateral and note the relation between them, as we change the positions of the points on the circle.

If the sum of the opposite sides of a quadrilateral are equal, can we draw a circle with the four sides as tangents?

Recall the fact we have seen earlier: in a quadrilateral formed by joining four points on a circle, the sum of the opposite angles are equal.

We have seen that among the lines drawn to a circle from a point, those which touch the circle at a single point are equal. We have also seen in the lesson, **Circles**, that for all lines cutting the circle at two points, the product of the whole line and the part outside the circle are equal.

Remember this picture and its equation?

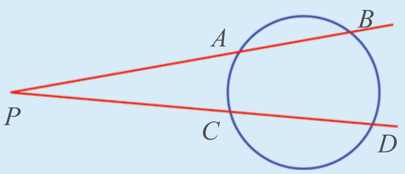


$$PA \times PB = PC \times PD$$

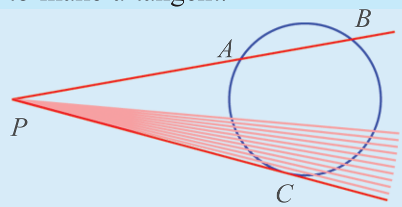
Now suppose, we draw a line which touches a circle and another line intersecting the circle.

Same relation

See this picture:



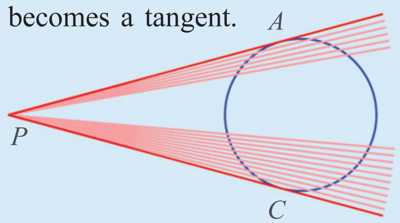
We have $PA \times PB = PC \times PD$. Suppose the lower line is rotated to make a tangent.



Then PD is the same as PC and the relation above becomes

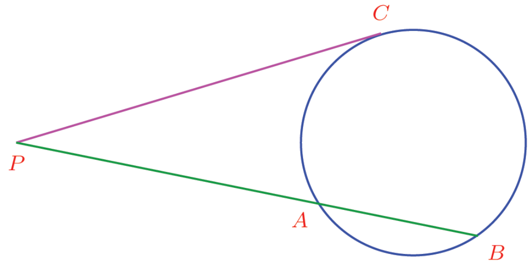
$$PA \times PB = PC^2$$

Suppose the upper line also becomes a tangent.

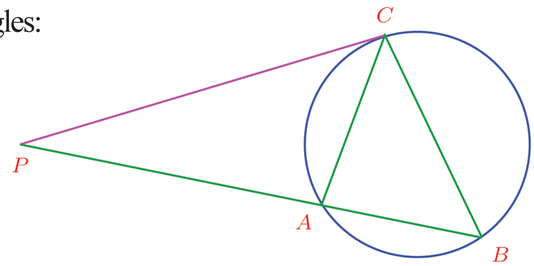


The relation becomes $PA^2 = PC^2$ or $PA = PC$.

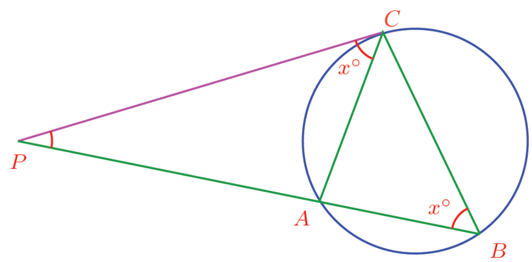
We have already seen that the length of the tangents from a point to a circle are equal.



To find out the relation between these, join AC, BC to make triangles:



The chord AC makes angle PCA at C; it is equal to the angle ABC, which AC makes on the other side of the circle, isn't it?



That is, the angle at C in ΔABC is equal to the angle at B in ΔPBC. And in both triangles the angle at P is the same.

Thus these triangles have the same angles and so pairs of sides opposite equal angles have the same ratio.

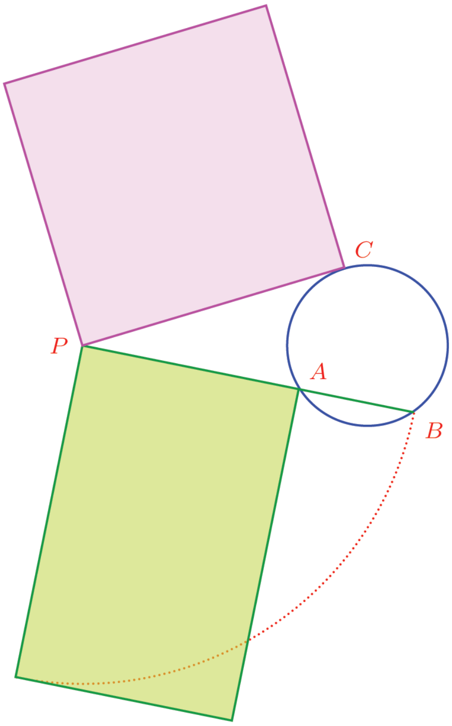
In ΔPAC, the side opposite the angle of x° is PA and in ΔPBC, the side opposite the angle of x° is PC. In ΔPAC, the longest side is PC and in ΔPBC, the longest side is PB. So,

$$\frac{PA}{PC} = \frac{PC}{PB}$$

We can write this as,

$$PA \times PB = PC^2$$

The product of an intersecting line and the part of it outside the circle is equal to the square of the tangent.

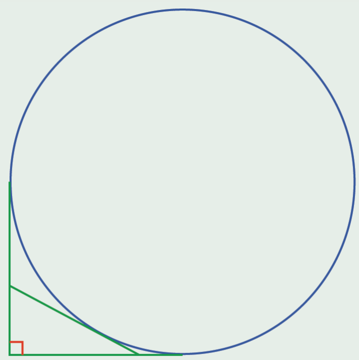


As in the case of intersecting chords, this can be stated in terms of areas.

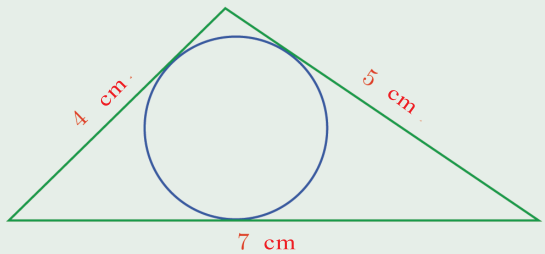
The rectangle with the intersecting line and its part outside the circle as sides and the square with sides equal to the tangent have the same area.

(1) In the picture, a triangle is formed by two mutually perpendicular tangents to a circle and a third tangent.

Prove that the perimeter of the triangle is equal to the diameter of the circle.

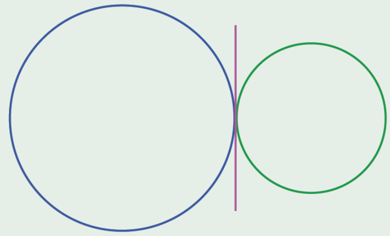


(2) The picture shows a triangle formed by three tangents to a circle.

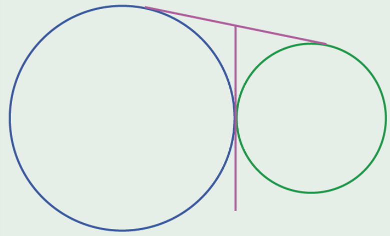


Calculate the length of each tangent from the corner of the triangle to the point of contact.

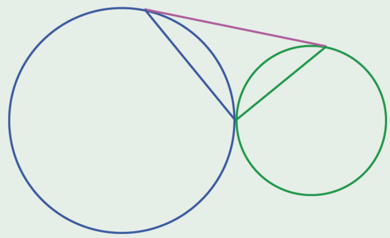
(3) In the picture, two circles touch at a point and the common tangent at this point is drawn.



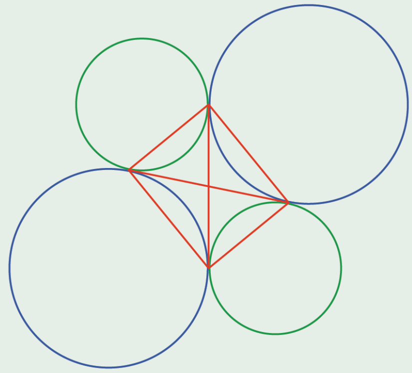
i) Prove that this tangent bisects another common tangent of these circles.



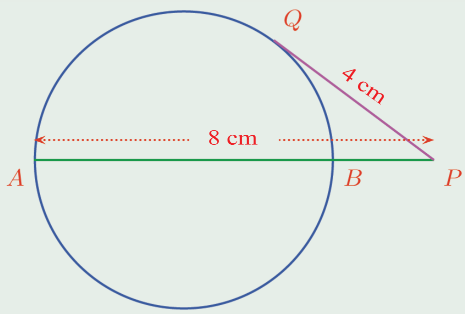
ii) Prove that the points of contact of these two tangents form the vertices of a right triangle.



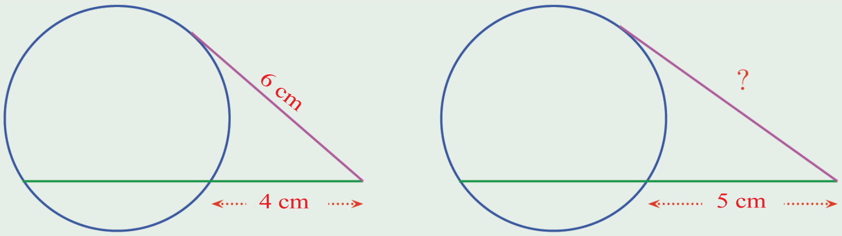
iii) Draw the picture on the right in your notebook, using convenient lengths.



(4) In the picture below, AB is a diameter and P is a point on AB extended. A tangent from P touches the circle at Q. What is the radius of the circle?



(5) In the first picture below, the line joining two points on a circle is extended by 4 centimetres and a tangent is drawn from this point. Its length is 6 centimetres, as shown:



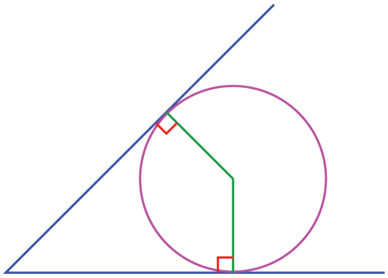
The second picture shows the same line extended by 1 centimetre more and a tangent drawn from this point. What is the length of this tangent?

(6) Draw a rectangle of one side 6 centimetres and area equal to that of a square of side 5 centimetres.

Circle touching a line

We have seen that from a point, two lines touching a circle can be drawn and also how we can draw these lines.

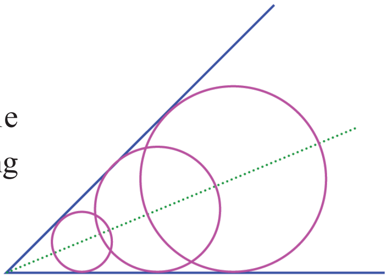
Now let's ask in reverse: can we draw a circle touching two lines meeting at a point? See this picture.



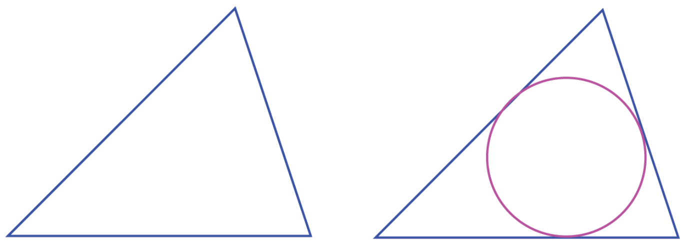
The radii are perpendicular to these lines. In other words, the centre of the circle must be at the same distance from these lines. So it must be on the bisector of this angle.

The centre of a circle touching two lines meeting at a point is on the bisector of the angle formed by the lines.

We can take any point on the angle bisector as centre to draw a circle touching the lines.

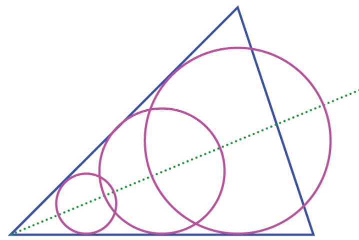


So, the next question is whether we can draw a circle touching all three sides of a triangle.

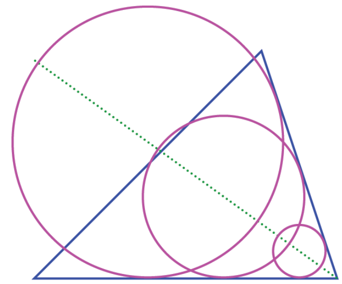


Draw an angle and its bisector in GeoGebra. Mark a point on this bisector and draw the perpendicular from this point to one of the sides of the angle. Mark their point of intersection. Draw the circle with centre at the point on the bisector and passing through the point on the side. Doesn't it touch both sides? Move the centre along the bisector and see.

We can take any point on the bisector of the angle made by the bottom and left side to draw a circle touching these two sides.



Taking any point on the bisector of the angle made by the bottom and right side as centre, we can draw a circle touching these two sides.

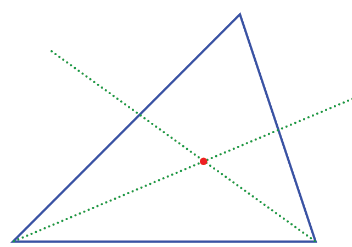


So, what if we take the point of both these bisectors, that is their point of intersection?

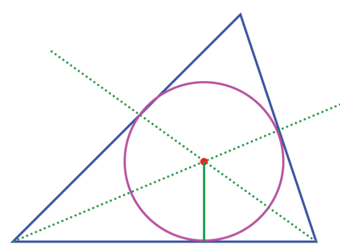
The lengths of the perpendiculars from this point to all three sides are equal, right?

What about the circle of radius this length, centred at this point?

This circle is called the *incircle* of the triangle.



We note another thing here. Since the centre of the incircle is at the same distance from the left and right sides, it is also on the bisector of the angle joining these sides.

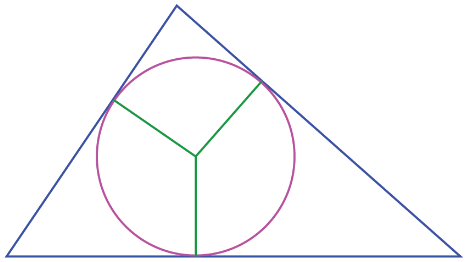


Draw a triangle and its incircle in GeoGebra.

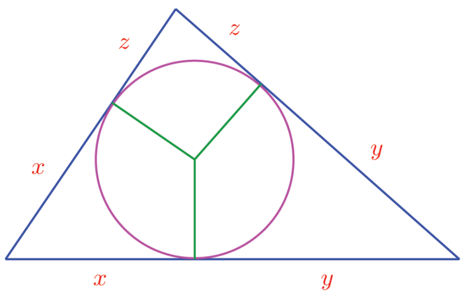
The bisectors of all three angles of a triangle meet at a point.

There are some relations between the points where the incircle touches the triangle and the sides of the triangle.

To see it, we draw the incircle of a triangle and join its centre to the points of contact with the sides.



The sides of the triangle are formed by the tangents to the incircle from the vertices. And the lengths of the tangents from each corner are equal. So taking the lengths of the tangents as x, y, z, we can mark them as below:

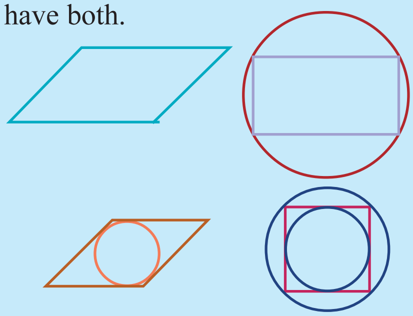


The sum of all these lengths is the perimeter of the triangle. That is, the perimeter of the triangle is $2(x+y+z)$. In other words, $x+y+z$ is half the perimeter of the triangle. Taking it as s,

$$x + y + z = s$$

Circumcircle and incircle

Every triangle has a circumcircle and an incircle. But in the case of quadrilaterals, some have neither of these, some have only one and some have both.



Next if we take the lengths of the sides of the triangle as a, b, c, the picture gives

$$x + y = a$$

$$y + z = b$$

$$z + x = c$$

Now to get x, we need only subtract $y+z$ from $x+y+z$:

$$x = (x+y+z) - (y+z) = s - b$$

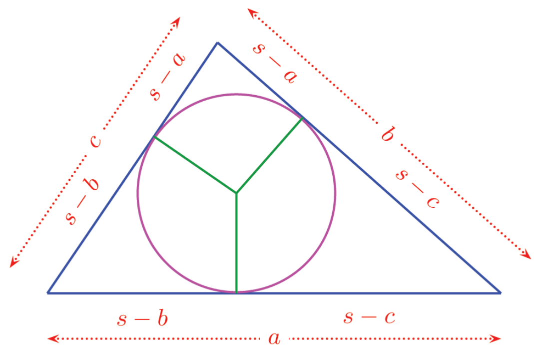
Similarly, we see that

$$y = (x+y+z) - (z+x) = s - c$$

and

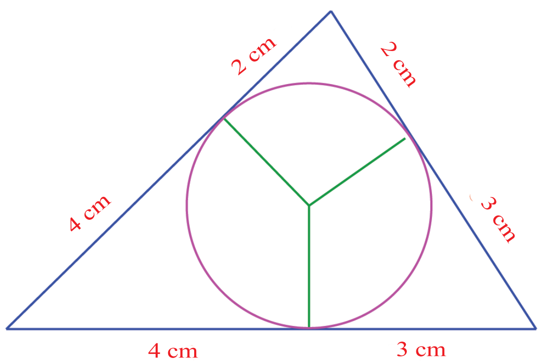
$$z = (x+y+z) - (x+y) = s - a$$

So the lengths of tangents are like this:

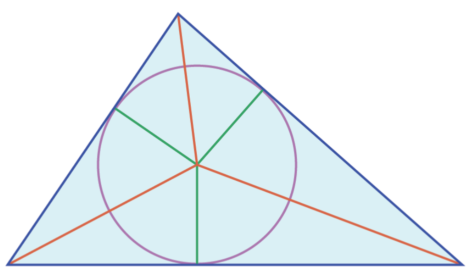


For example, consider a triangle of sides 5 centimetres, 6 centimetres and 7 centimetres. Half the perimeter is 9 centimetres.

So the points of contact with the incircle divides the sides like this: $9 - 5 = 4$, $9 - 6 = 3$, $9 - 7 = 2$



The radius of the incircle has a relation with the area of the triangle. The lines joining the centre of the incircle to the vertices divide the triangle into three.



One side of each of these small triangles is a side of the original large triangle and the height from it is equal to the radius of the incircle. So, if we take the sides of the triangle as a, b, c and the radius of the incircle as r, then the areas of these small triangles are $\frac{1}{2}ar$, $\frac{1}{2}br$, $\frac{1}{2}cr$.

Their sum is the whole area of the large triangle. Taking it as A, we have

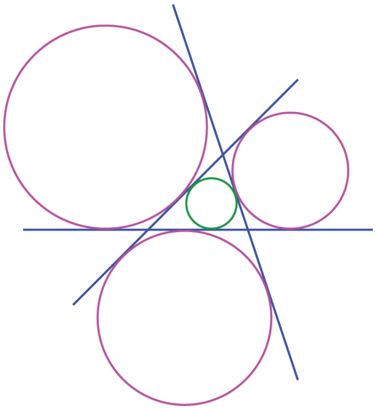
$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a+b+c)\,r = sr$$

This equation can written as

$$r = \frac{A}{s}$$

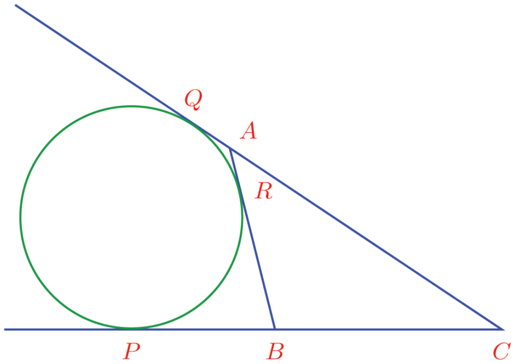
The radius of the incircle of a triangle is its area divided by half the perimeter.

The incircle of a triangle is the circle inside the triangle, which touches all three sides. If the requirement is only touching all three sides, then we can draw three more such circles.



These are called *excircles* of the triangle. They are drawn using the bisectors of the external angles.

Consider a triangle and one of its excircles:



P, Q, R are the points where the circle touches the sides of the triangle.

Let's find the lengths of the tangents CP and CQ. Taking the length BC, CA, AB of the triangle as a, b, c, we get

$$CP = CB + BP = a + BP \qquad CQ = CA + AQ = b + AQ$$

Now $BP = BR$, being tangents from B and $AQ = AR$, being tangents from A. So,

$$CP = a + BR \qquad CQ = b + AR$$

We also see that $AR + RB = AB$. This together with the two equations above gives

$$CP + CQ = a + b + BR + AR = a + b + c$$

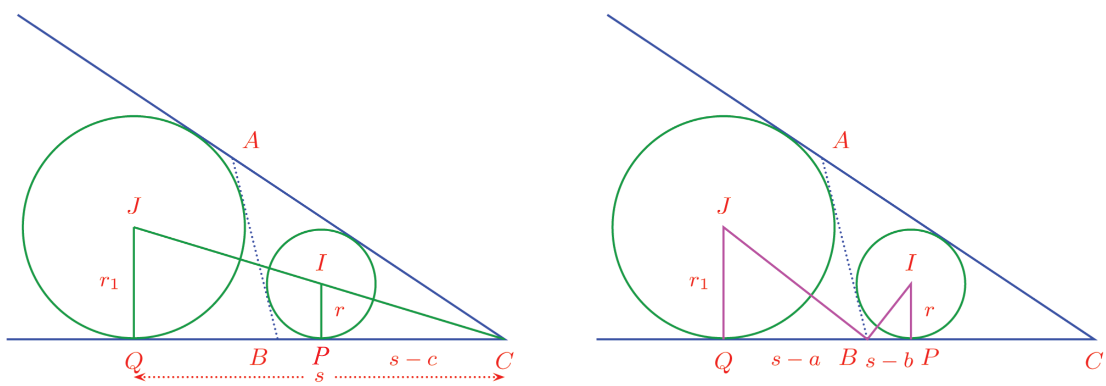
But this is the perimeter of the triangle. Also, CP and CQ have the same length. So, what do we get?

$$CP = CQ = s$$

Thus we have the following:

> The lengths of the tangents from a vertex of the triangle to the excircle opposite it are equal to the perimeter of the triangle.

Now let's draw the incircle also. Let r be the radius of the incircle and r_1, the radius of the excircle.



ΔCIP and ΔCJQ of the picture on the left have the same angles, and so ratio of sides opposite equal angles are also the same.

(1) $$\frac{r}{r_1} = \frac{s - c}{s}$$

Now in the picture on the right, look at the angles of ΔBIP and ΔBJQ at B. Since BI, BJ are the bisectors of the internal and external angles at B of ΔABC,

$$\angle QBJ = \frac{1}{2}\angle QBA = \frac{1}{2}(180^\circ - \angle CBA) = 90^\circ - \angle PBI$$

Thus ΔPBI and ΔQBJ have the same angles, which gives

$$\frac{r}{s-a} = \frac{s-b}{r_1}$$

Using cross multiplication, we get

(2) $$rr_1 = (s-a)(s-b)$$

Now from equations (1) and (2), we have

$$\frac{r}{r_1} \times rr_1 = \frac{s-c}{s} \times (s-a)(s-c)$$

That is,

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}$$

We have already noted that the area of ΔABC is rs. Using the above equation the area is

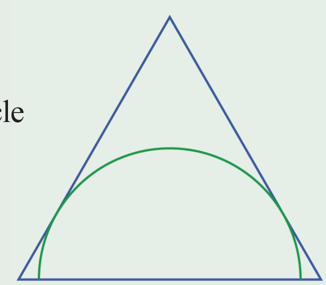
$$\sqrt{s(s-a)(s-b)(s-c)}$$

This method of computing the area of a triangle using the lengths of its side is known as Heron's Formula.

(1) Draw a triangle of sides 4 centimetres, 5 centimetres, 6 centimetres and draw its incircle. Calculate its radius.

(2) Draw a rhombus of sides 5 centimetres and one angle 50° and draw its incircle.

(3) Draw an equilateral triangle and a semicircle touching its two sides, as in the picture.



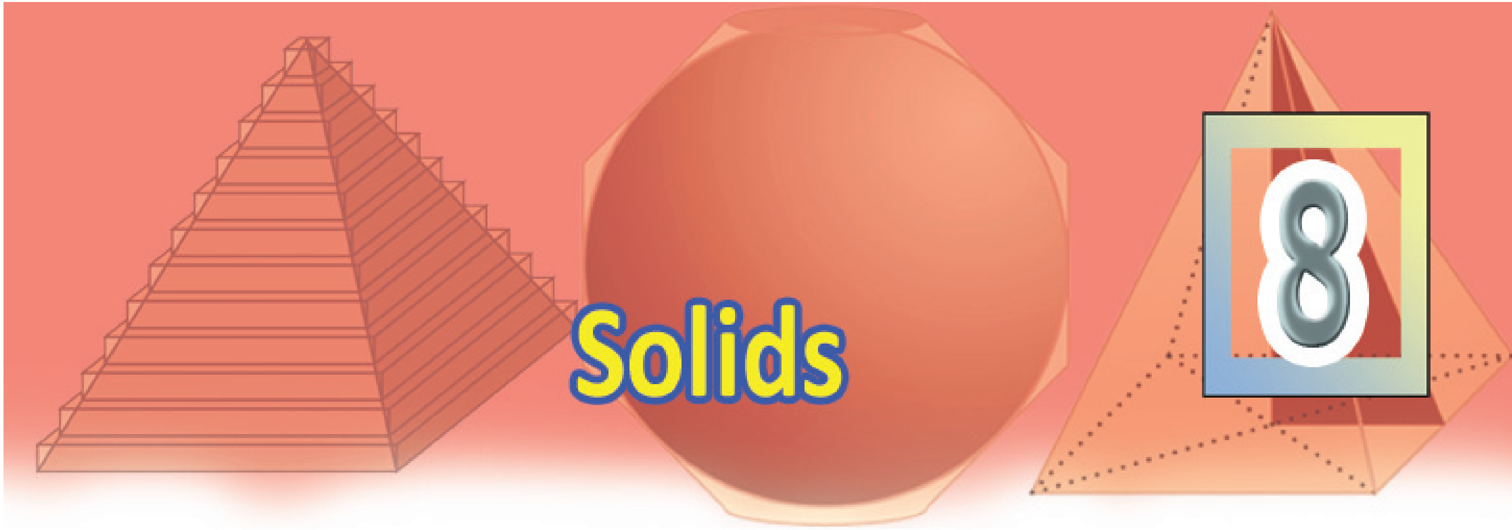
(4) Prove the radius of the incircle of an equilateral triangle is half the radius of its circumcircle.

(5) Prove that if the hypotenuse of a right triangle is h and the radius of its incircle is r, then its area is $r(h+r)$.

(6) Calculate the area of a triangle of sides 13 centimetres, 14 centimetres, 15 centimetres.

Looking back

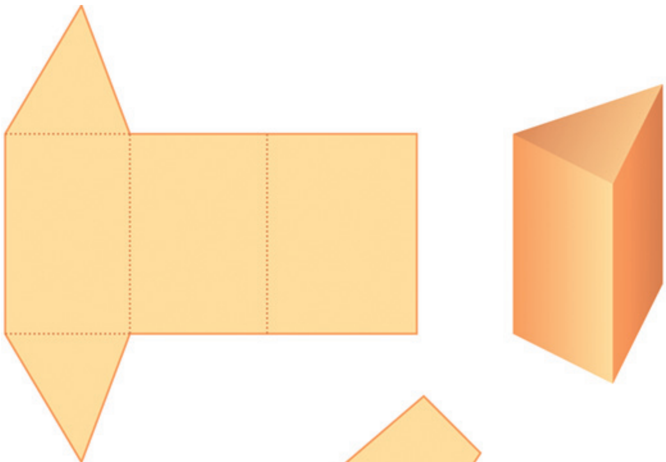
Learning outcomes	On my own	With teacher's help	Must improve
• Understanding what happens to a line joining two points on a circle when they are closer and closer, and hence the idea of a tangent.			
• Realising that the tangent at a point on a circle and the radius through the point are perpendicular to each other.			
• Understanding the relations between the angle between the tangents to a circle at the ends of a chord, the central angle of the chord and the angle which the chord makes at a point on the circle.			
• Realising that two tangents can be drawn to a circle from a point outside and actually drawing them.			
• Drawing a circle touching all three sides of a triangle.			

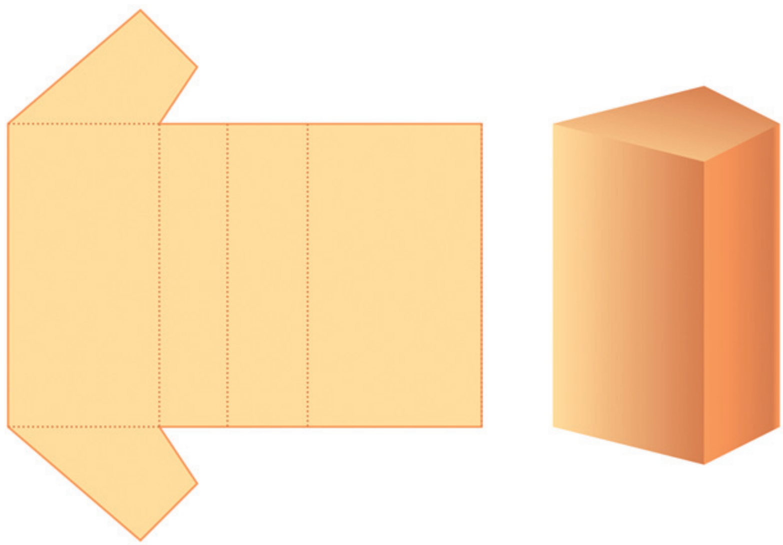


8 Solids

Pyramids

We can make prisms by cutting thick paper in various ways and pasting the edges.





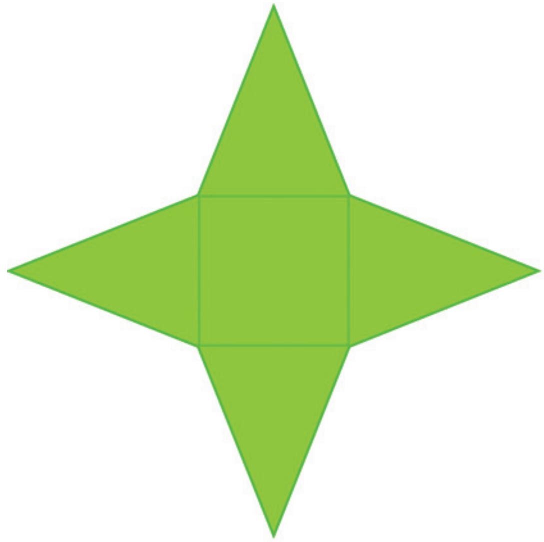
And we have learnt much about them.

Let's make another kind of solid:

Pyramids in GeoGebra

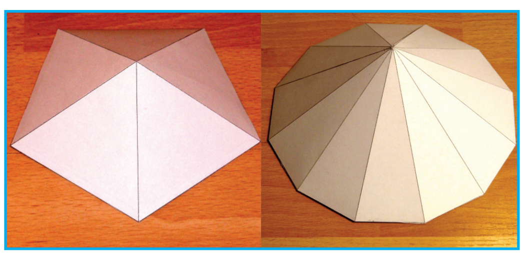
We have seen in Class 9 how prisms can be drawn in GeoGebra. Now let's see how pyramids can be drawn. Open 3D Graphics and make the necessary preparations (See the section, **Solids in GeoGebra** of the lesson, **Prisms** in the Class 9 textbook) Draw a square in Graphics. Choose **Extrude to Pyramid or Cone** in 3D Graphics and click on the square. In the window opening up, type the height of the pyramid. (we can also make a slider and give its name as the height).

First, cut out a figure like this in paper.



A square in the middle and four triangles around it; all four of them are isosceles triangles and they are equal. Now fold and paste as shown below.

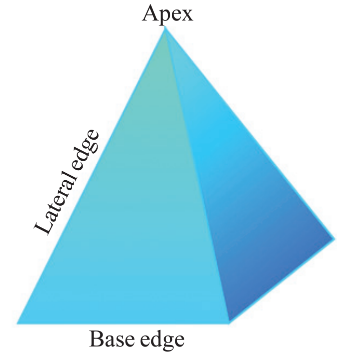
What shape is this? Can't be called a prism; prisms have two equal bases and rectangles on the sides. In the shape we have made now, we have a square at the bottom, a point on top and triangles all around.



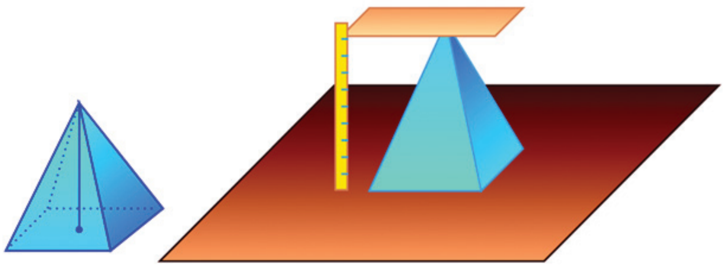
Instead of square, the base can be some other rectangle, a triangle or some other polygon. Try! (It is better looking when the base is a regular polygon.)

Such a solid is generally called a *pyramid*.

The sides of the polygon forming the base of a pyramid are called *base edges* and the other sides of the triangles are called *lateral edges*. The topmost point of a pyramid is called its *apex*.

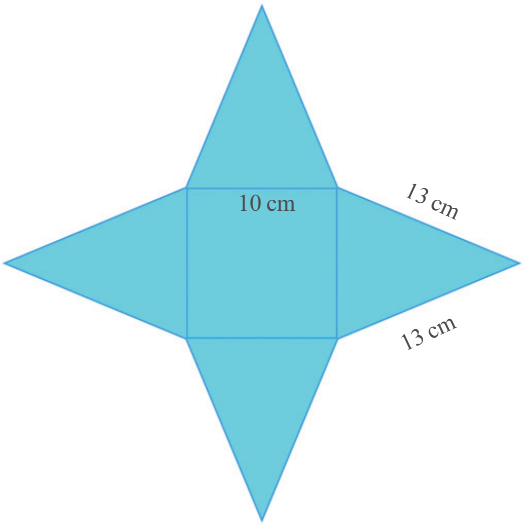


The height of a prism is the distance between its bases, isnt it? The height of a pyramid is the perpendicular distance from the apex to the base.



Area

What is the surface area of a square pyramid of base edges 10 centimetres and lateral edges 13 centimetres? Surface area is the area of paper needed to make it. How will it look, if we cut this pyramid open and lay it flat?



Let's see how GeoGebra helps us to see the 'cut and spread' shape of a pyramid. Make a pyramid in 3D graphics as described earlier. Choose **Net** and click on the pyramid. We get the shape of the paper used to make it (It is called the net of the solid). We also get a slider in **Graphics**. By moving the slider, we can see how the pyramid is made from the net. We can also hide the original pyramid by clicking against the pyramid in the **Algebra** window.

The area of the square is easily seen to be 100 square centimetres. What about the triangles?

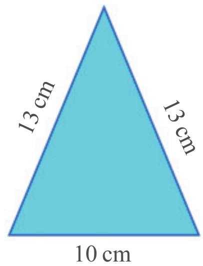
The sides of the triangle are 10, 13, 13 centimetres. We have Heron's help to compute the areas; subtracting each side from half the perimeter, we can calculate the area as

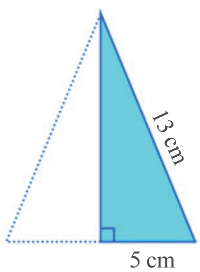
$$\sqrt{18 \times 8 \times 5 \times 5} = \sqrt{9 \times 16 \times 5 \times 5} = 60$$

Thus the area of each triangular face is 60 square centimetres. So the surface area of the pyramid is,

$$100 + (4 \times 60) = 340 \text{ square centimetres}$$

The area of the triangle can also be computed as half the product of its base and height.





For that, we need the height of the triangle. Since the triangle is isosceles, this altitude bisects the base.

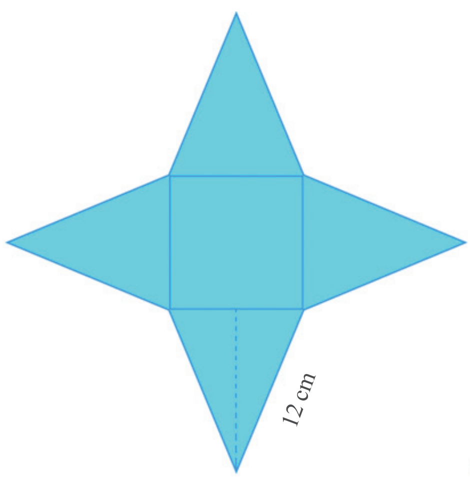
So using Pythagoras Theorem, the height of the triangle is

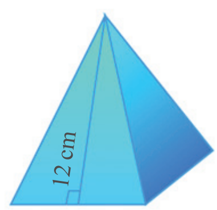
$$\sqrt{13^2 - 5^2} = 12 \text{ centimetres}$$

Height and slant height

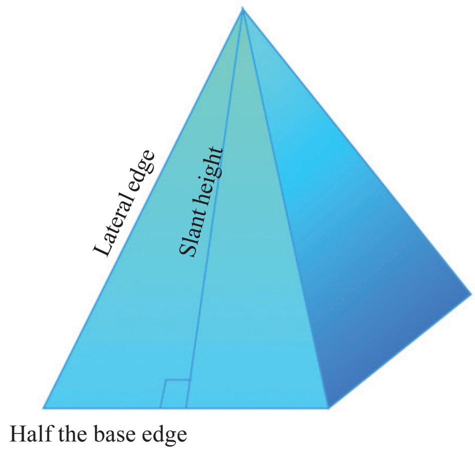
Draw a pyramid in GeoGebra. Click **Midpoint or Centre** to mark the mid points of a base edge and the midpoint of a diagonal of the base. Use **Segment** to mark the height and slant height of the pyramid. Use **Polygon** to make the right triangle with height, slant height and half the base edge as sides. Using **Net**, the pyramid can be cut and spread. We can also hide the pyramid.

Thus the area of the triangle is 60 square centimetres. What will be the height of the triangle, when the paper is turned into a pyramid?



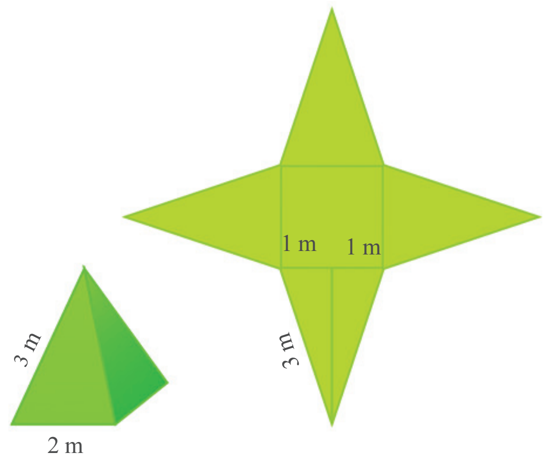


This length is called the *slant height* of the pyramid.



We have seen the relation between the base edge, lateral edge and slant height of a pyramid in the problem we did just now. As shown in the picture on the right, there is a right triangle on each side of the pyramid - its perpendicular sides are the slant height and half the base edge, the hypotenuse is a lateral edge.

Now do this problem: what is the surface area of a pyramid with base edges 2 metres and lateral edges 3 metres?



The base area is 4 square metres. To compute the areas of lateral faces, we need the slant height. In the right triangle mentioned above, one side is half the base edge, that is, 1 metre and the hypotenuse is the lateral edge of 3 metres. So, slant height is

$$\sqrt{3^2 - 1^2} = 2\sqrt{2} \text{ metres}$$

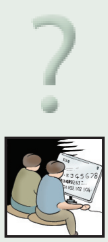
Using this, the area of each triangular face is

$$\frac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2} \text{ square metres.}$$

So, the surface area of the pyramid is,

$$4 + \left(4 \times 2\sqrt{2}\right) = 4 + 8\sqrt{2} \text{ square metres.}$$

If not satisfied with this, a calculator can be used, (or an approximate value of $\sqrt{2}$ recalled) to compute this as 15.1 square metres.



(1) A square of side 5 centimetres, and four isosceles triangles of base 5 centimetres and height 8 centimetres, are to be put together to make a square pyramid. How many square centimetres of paper is needed?

(2) A toy is in the shape of a square pyramid of base edge 16 centimetres and slant height 10 centimetres. What is the total cost of painting 500 such toys, at 80 rupees per square metre?

(3) The lateral faces of a square pyramid are equilateral triangles and the length of a base edge is 30 centimetres. What is its surface area?

(4) The perimeter of the base of square pyramid is 40 centimetres and the total length of all its edges is 92 centimetres. Calculate its surface area.

(5) Can we make a square pyramid with the lateral surface area equal to the base area?

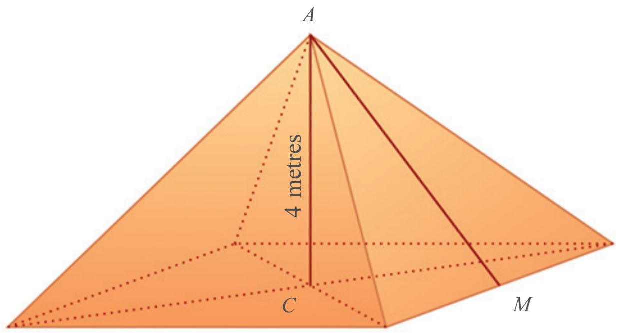
Height and slant height

The height of a pyramid is often an important measure. See this problem:

> A tent is to be made in the shape of a square pyramid of base edges 6 metres and height 4 metres. How many square metres of canvas is needed to make it?

To calculate the area of the triangular faces of the tent, we need the slant height. How do we compute it using the given specifications?

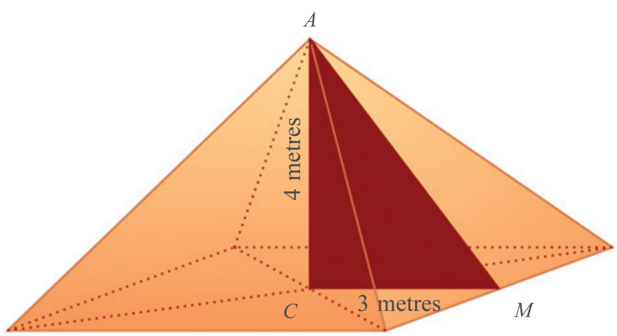
See this picture:



The slant height we need is AM. Joining CM, we get a right triangle with AM as hypotenuse. What is the length of CM in it?

Pyramids of Egypt

The very word pyramid brings to our mind the great pyramids of Egypt. 138 such pyramids are found from various parts of Egypt. Many of them were built around 2000 BC.

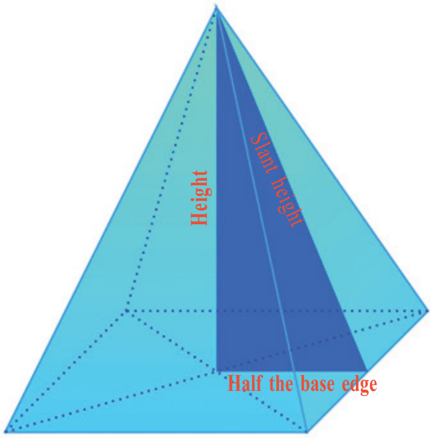


From the picture, $AM = \sqrt{3^2 + 4^2} = 5$ metres.

So to make the tent, four isosceles triangles of base 6 metres and height 6 metres are needed. Their total area is $4 \times \frac{1}{2} \times 6 \times 5 = 60$ square metres.

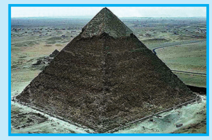
So this much canvas is needed to make the tent.

In this problem, we have found something which is true in the case of all square pyramids. Within every square pyramid, we can imagine a right triangle with perpendicular sides as the height of the pyramid and half the base edge and hypotenuse as the slant height.



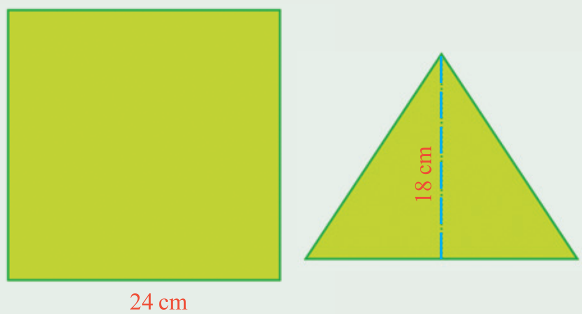
Great Pyramid

The largest pyramid in Egypt is the Great Pyramid of Giza.



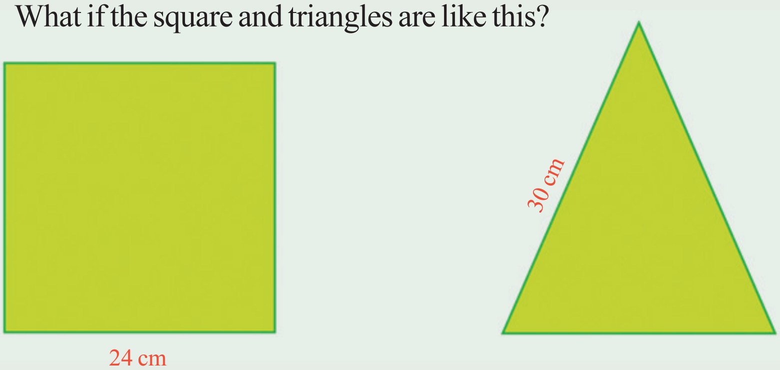
Its base is a square of almost half a lakh square metres and its height is about 140 metres. It is estimated that about 20 years would have been needed to complete it. These royal tombs built with huge blocks of stones stacked with precision to end in a point are living symbols of human labour, engineering skill and mathematical knowledge.

(1) Using a square and four triangles with dimensions as specified in the picture, a pyramid is made.



What is the height of this pyramid?

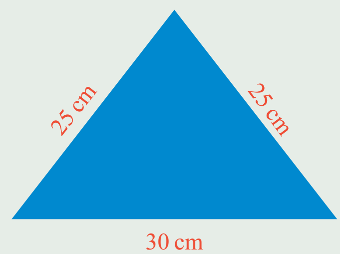
What if the square and triangles are like this?



(2) A square pyramid of base edge 10 centimetres and height 12 centimetres is to be made of paper. What should be the dimensions of the triangles?

(3) Prove that in any square pyramid, the squares of the height, slant height and lateral edge are in arithmetic sequence.

(4) A square pyramid is to be made with the triangle shown here as a lateral face. What would be its height? What if the base edge is 40 centimetres instead of 30 centimetres?

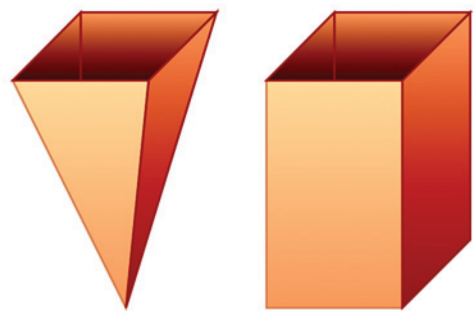


Can we make a square pyramid with any four equal isosceles triangles?

Volume of a pyramid

We have seen that the volume of any prism is equal to the product of the base area and the height. What about the volume of a pyramid?

Let's take the case of a square pyramid. Make a hollow square pyramid with thick paper and also a square prism of the same base and height.



Draw a square pyramid and square prism of the same base and with the same height in GeoGebra. To distinguish between them, change the colour of the pyramid and make Opacity 100. (**Object properties** → **Colour**). Find their volumes using **Volume.** What is the relation between them? Change the base and height and see.

Fill the pyramid with sand and transfer it to the prism. Measure the height of the sand in the prism and see what fraction of the height of the prism it is. A third isn't it? So to fill the prism, how many times should we fill the pyramid?

Thus we see that the volume of the prism is three times the volume of the pyramid. (A mathematical explanation of this is given at the end of this lesson).

We have seen in Class 9 that the volume of a prism is equal to the product of the base area and the height.

So what can we say about the volume of a square pyramid?

The volume of a square pyramid is equal to a third of the product of the base area and the height.

For example, the volume of a square pyramid of base edge 10 centimetres and height 8 centimetres is $\frac{1}{3} \times 10^2 \times 8 = 266\frac{2}{3}$ cubic centimetres.

A metal cube of edges 15 centimetres is melted and recast into a square pyramid of base edge 25 centimetres. What is its height?

The volume of the cube is 15^3 cubic centimetres. The volume of the square pyramid is also this. And the volume of a pyramid is a third of the product of the base area and height. Since the base area of our pyramid is 25^2 square centimetres, a third of the height is $\frac{15^3}{25^2}$ and so the height is,

$$3 \times \frac{15^3}{25^2} = 16.2 \text{ centimetres.}$$

(1) What is the volume of a square pyramid of base edge 10 centimetres and slant height 15 centimetres?

(2) Two square pyramids have the same volume. The base edge of one is half that of the other. How many times the height of the second pyramid is the height of the first?

(3) The base edges of two square pyramids are in the ratio 1 : 2 and their heights in the ratio 1 : 3. The volume of the first is 180 cubic centimetres. What is the volume of the second?

(4) All edges of a square pyramid are 18 centimetres. What is its volume?

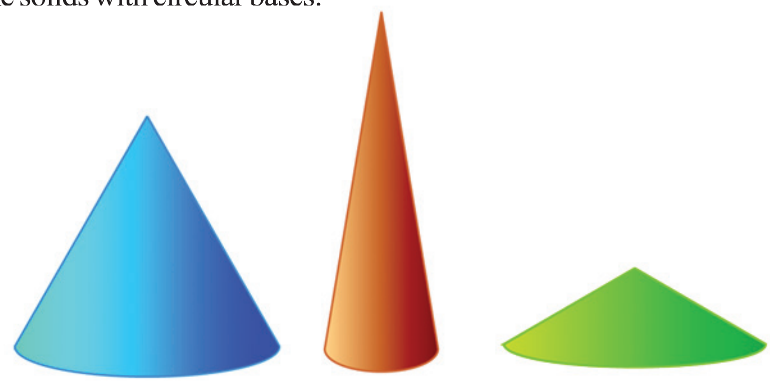
(5) The slant height of a square pyramid is 25 centimetres and its surface area is 896 square centimetres. What is its volume?

(6) All edges of a square pyramid are of the same length and its height is 12 centimetres. What is its volume?

(7) What is the surface area of a square pyramid of base perimeter 64 centimetres and volume 1280 cubic centimetres?

Cone

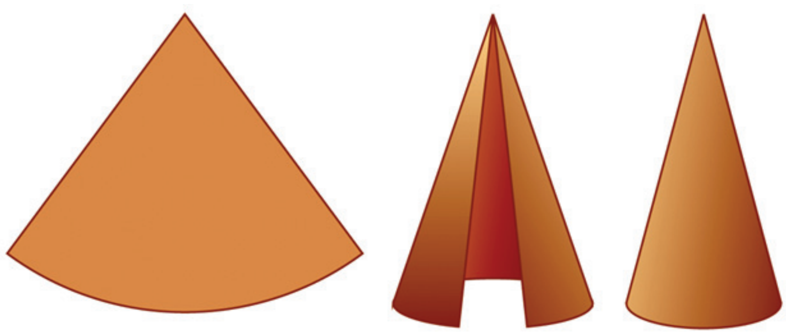
Cylinders are prism-like solids with circular bases. Similarly, we have pyramid-like solids with circular bases:



They are called *cones*.

We can make a cylinder by rolling up a rectangle. Likewise, we can make a cone by rolling up a sector of a circle.

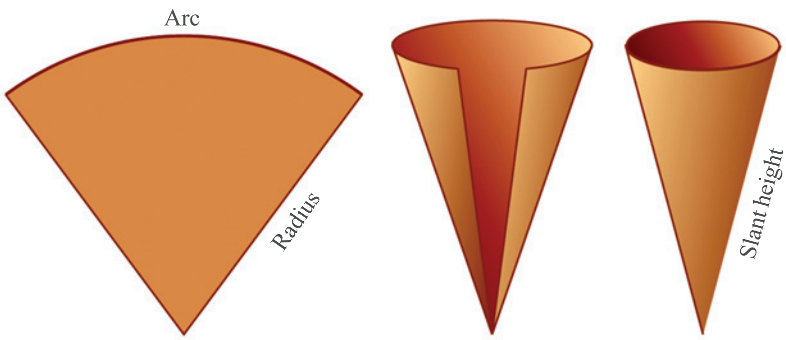
What is the relation between the dimensions of the sector we start with and the cone we end up with?



The radius of the sector becomes the slant height of the cone. The arc length of the sector becomes the circumference of the base of the cone.

Cone

We can draw cones in GeoGebra, just as we drew pyramids. Draw a circle in **Graphic** and in **3D Graphics**, use **Extrude to Pyramid or Cone**. Base radius and height can be changed using sliders.



We often specify the size of a sector in terms of the central angle. See this problem:

> From a circle of radius 12 centimetres, a sector of central angle 45° is cut out and made into a cone. What are the slant height and base radius of this cone?

Slant height of the cone is the radius of the circle itself: 12 centimetres. What about its base radius?

45° is $\frac{1}{8}$ of 360°. And the arc length of a sector is proportional to the central angle. So this arc length is $\frac{1}{8}$ of the circumference of the full circle.

This arc becomes the base circle of the cone. Thus the circumference of the base circle of the cone is $\frac{1}{8}$ of the circumference of the larger circle from which the sector was cut out. Since radii of circles are proportional to their circumferences, the radius of the smaller circle is $\frac{1}{8}$ of the radius of the large circle. Thus the radius of the base of the cone is $\frac{1}{8} \times 12 = 1.5$ centimetres.

How about a question in the reverse direction?

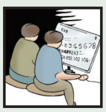
> How do we make a cone of base radius 5 centimetres and slant height 15 centimetres?

To make a cone, we need a sector. Since the slant height is to be 15 centimetres, the sector must be cut out from a circle of radius 15 centimetres.

What should be its central angle?

The radius of the small circle forming the base of the cone is $\frac{5}{15} = \frac{1}{3}$ of the radius of the large circle from which the sector is to be cut out. (How do we get this?). So, the circumference of the small circle is also $\frac{1}{3}$ of the circumference of the large circle.

The circumference of the small circle is the arc length of the sector. Thus the arc of the sector is $\frac{1}{3}$ of the circle from which it is cut out. So its central angle must be $360 \times \frac{1}{3} = 120°$.



(1) What are the radius of the base and slant height of a cone made by rolling up a sector of central angle 60° cut out from a circle of radius 10 centimetres?

(2) What is the central angle of the sector to be used to make a cone of base radius 10 centimetres and slant height 25 centimetres?

(3) What is the ratio of the base-radius and slant height of a cone made by rolling up a semicircle?

Curved surface area

As in the case of a cylinder, a cone also has a curved surface - the part which rises up at a slant. The area of this curved surface is the area of the sector used to make the cone. (For a cylinder also, the area of the curved surface is the area of the rectangle rolled up to make it, isn't it?)

See this problem:

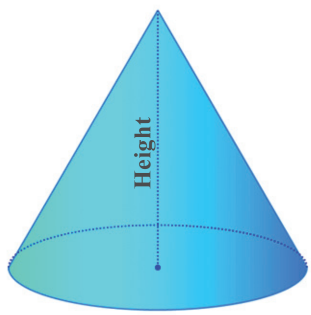
> To make a conical hat of base radius 8 centimetres and slant height 30 centimetres, how much square centimetres of paper do we need?

What we need here is the area of the circular sector we roll up to make this hat. Since the slant height is to be 30 centimetres, we must cut out the sector from a circle of this radius. Also the radius of the small circle forming the base of the cone must be 8 centimetres, that is $\frac{8}{30} = \frac{4}{15}$ of the radius of the large circle from which the sector is cut out. So the circumference of small circle is also the same fraction of the circumference of the large circle. The arc length of the sector is the circumference of the small circle. Thus the sector to be cut out is $\frac{4}{15}$ of the full circle. So, its area is the fraction of the area of the circle; that is,

$$\pi \times 30^2 \times \frac{4}{15} = \pi \times 2 \times 30 \times 4 = 240\pi$$

Thus we need 240π square centimetres of paper to make the hat (It can be computed as approximately 754 square centimetres).

As in a pyramid, the height of a cone is the perpendicular distance from the apex to the base, and it is the distance between the apex and the centre of the base circle.



Curved surface

The area of the curved surface of a cone is the area of the sector used to make it. If we take the base radius of the cone as r and its slant height as l, then the radius of the sector is l and its central angle is $\frac{r}{l} \times 360°$. So its area is

$$\frac{1}{360} \times \left(\frac{r}{l} \times 360\right) \times \pi l^2 = \pi r l$$

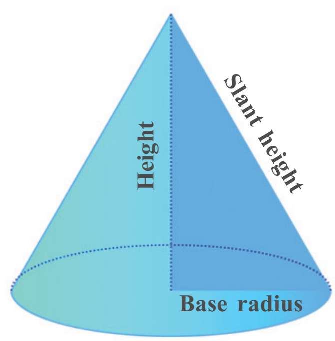
(Recall the computation of the area of a sector in Class 9).

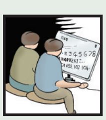
Thus the area of the curved surface of a cone is half the product of the base circumference and the slant height.

Again as in the case of a square pyramid, the height is related to the slant height via a right triangle.

For example, in a cone of base radius 5 centimetres and height 10 centimetres the slant height is,

$$\sqrt{5^2 + 10^2} = \sqrt{125} = 5\sqrt{5} \text{ centimetres}$$





(1) What is the area of the curved surface of a cone of base radius 12 centimetres and slant height 25 centimetres?

(2) What is the surface area of a cone of base diameter 30 centimetres and height 40 centimetres?

(3) A conical fire work is of base diameter 10 centimetres and height 12 centimetres. 10000 such fireworks are to be wrapped in colour paper. The price of the colour paper is 2 rupees per square metre. What is the total cost?

(4) Prove that for a cone made by rolling up a semicircle, the area of the curved surface is twice the base area.

Volume of a cone

To find the volume of a cone, we can do an experiment similar to the one we did to find the volume of a square pyramid. Make a cone and a cylinder of the same base and height. Fill the cone with sand and transfer it to the cylinder. Here also, we can see that the volume of the cone is a third of the volume of the cylinder. Thus we have the following:

The volume of a cone is equal to a third of the product of the base area and height.

As in the case of square pyramids, draw a cylinder and a cone of same base and height in GeoGebra. Compare their volumes.

(A mathematical explanation of this also is given at the end of this lesson)

For example, the volume of a cone of base radius 4 centimetres and height 6 centimetres is

$$\frac{1}{3} \times \pi \times 4^2 \times 6 = 32\pi \text{ cubic centimetres.}$$

(1) The base radius and height of a cylindrical block of wood are 15 centimetres and 40 centimetres. What is the volume of the largest cone that can be carved out of this?

(2) The base radius and height of a solid metal cylinder are 12 centimetres and 20 centimetres. By melting it and recasting, how many cones of base radius 4 centimetres and height 5 centimetres can be made?

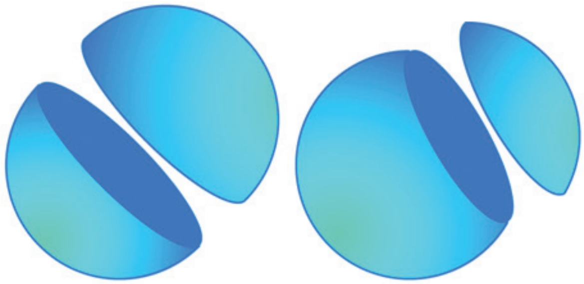
(3) A sector of central angle 216° is cut out from a circle of radius 25 centimetres and is rolled up into a cone. What are the base radius and height of the cone? What is its volume?

(4) The base radii of two cones are in the ratio 3 : 5 and their heights are in the ratio 2 : 3. What is the ratio of their volumes?

(5) Two cones have the same volume and their base radii are in the ratio 4 : 5. What is the ratio of their heights?

Sphere

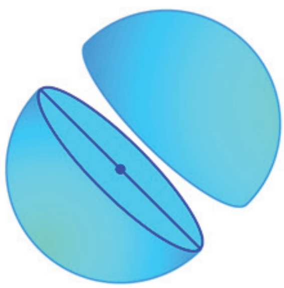
Round solids enter our lives in various ways - as the thrill of ball games and as the sweetness of laddus. Now let's look at the mathematics of such solids called *spheres*.



It we slice cylinder or cone parallel to a base, we get a circle. In whatever way we slice a sphere, we get a circle.

The distance of any point on a circle from the centre is the same. A sphere also has a *centre*, from which the distance to any point on its surface is the same. This distance is called the *radius* of the sphere and double this is called the *diameter*.

If we slice a sphere into exact halves, we get a circle whose centre, radius and diameter are those of the sphere itself.



We cannot cut open a sphere and spread it flat, as we did with other solids. The fact is that we cannot make the surface of a sphere flat without some folding or stretching.

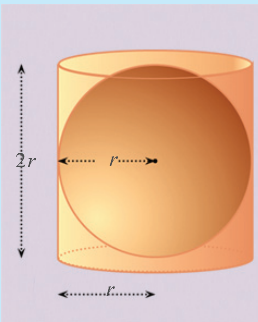
But we can prove that the surface area of a sphere of radius r is $4\pi r^2$ (An explanation is given at the end of the lesson).

The surface area of a sphere is equal to the square of its radius multiplied by 4π.

Also, we can prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ (An explanation of this also is given at the end of the lesson.)

Sphere and cylinder

Consider a cylinder which can cover a sphere precisely. Its base radius is the radius of the sphere and its height is double its radius.



So if we take the radius of the sphere as r, the base radius and height of the cylinder are r and $2r$.

So its surface area is

$$(2\pi r \times 2r) + (2 \times \pi r^2) = 6\pi r^2$$

The surface area of the sphere is $4\pi r^2$. Thus the ratio of these surface areas is 3 : 2.

Again, the volume of the cylinder is

$$\pi r^2 \times 2r = 2\pi r^3$$

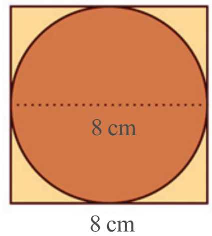
and the volume of the sphere is $\frac{4}{3}\pi r^3$, so that the ratio of the volumes is also 3 : 2

See this problem:

What is the surface area of the largest sphere that can be carved from a cube of edges 8 centimetres?

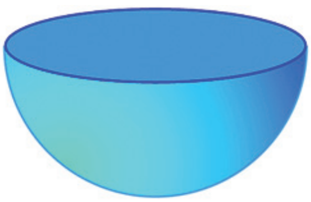
We can see from the picture that the diameter of the sphere is the length of an edge of the cube. So, the surface area of the sphere is

$4\pi \times 4^2 = 64\pi$ square centimetres



Another problem:

A solid sphere of radius 12 centimetres is cut into two equal halves. What is the surface area of each hemisphere?



The surface of the hemisphere consists of half the surface of the sphere and a circle.

Since the radius of the sphere is 12 centimetres, its area is,

$$4\pi \times 12^2 = 576\pi \text{ square centimetres}$$

Since the radius of the circle is 12 centimetres, its area is

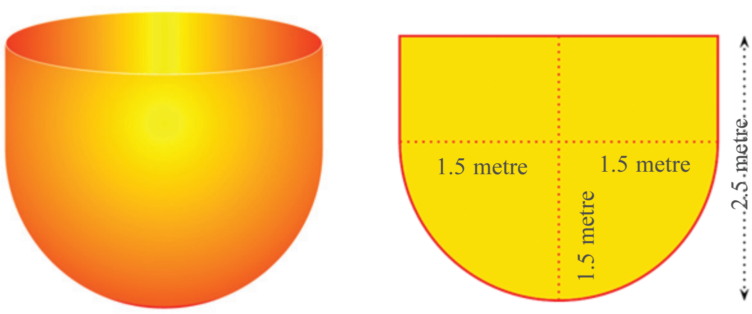
$$\pi \times 12^2 = 144\pi \text{ square centimetres}$$

So the surface area of the hemisphere is,

$$\frac{1}{2} \times 576\pi + 144\pi = 432\pi \text{ square centimetres}$$

One more example:

A water tank is in the shape of a hemisphere attached to a cylinder. Its radius is 1.5 metres and the total height is 2.5 metres. How many litres of water can it hold?



The volume of the hemispherical part of the tank is,

$$\frac{2}{3}\pi \times (1.5)^3 = 2.25\pi \text{ cubic metres.}$$

And the volume of the cylindrical part is

$$\pi \times (1.5)^2 (2.5 - 1.5) = 2.25\pi \text{ cubic metres.}$$

So, the total volume is

$$2.25\pi + 2.25\pi = 4.5\pi \approx 14.13 \text{ cubic metres.}$$

Since one cubic metre is 1000 litres, the tank can hold about 14130 litres of water.

(1) The surface area of a solid sphere is 120 square centimetres. If it is cut into two halves, what would be the surface area of each hemisphere?

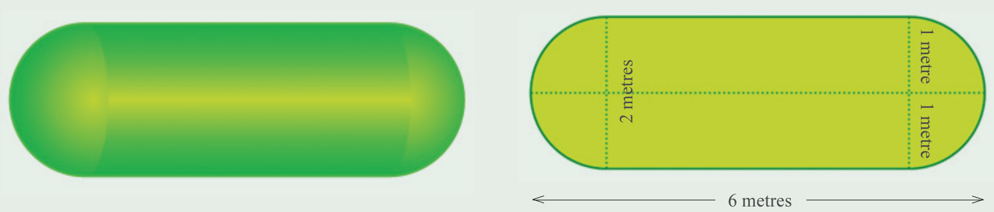
(2) The volume of two spheres are in the ratio 27 : 64. What is the ratio of their radii? And the ratio of their surface areas?

(3) The base radius and length of a metal cylinder are 4 centimetres and 10 centimetres. If it is melted and recast into spheres of radius 2 centimetres, how many spheres can be made?

(4) A metal sphere of radius 12 centimetres is melted and recast into 27 small spheres. What is the radius of each sphere?

(5) From a solid sphere of radius 10 centimetres, a cone of height 16 centimetres is carved out. What fraction of the volume of the sphere is the volume of the cone?

(6) The picture shows the dimensions of a petrol tank.



How many litres of petrol can it hold?

(7) A solid sphere is cut into two hemispheres. From one, a square pyramid and from the other a cone, each of maximum possible size are carved out. What is the ratio of their volumes?

What is the speciality of the square pyramid of maximum volume that can be cut out from a solid hemisphere?

Looking back

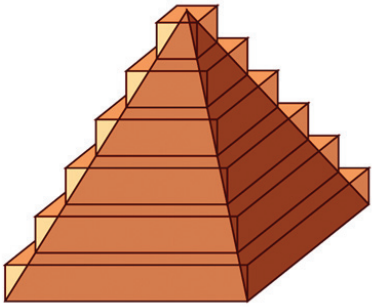
Learning outcomes	On my own	With teacher's help	Must improve
• Finding the relation between the base edge, lateral edge, height, slant height and base diagonal of a square pyramid.			
• Making square pyramids of specified dimensions by cutting out squares and isosceles triangles.			
• Computing the surface area and volume of a square pyramid using given measurements.			
• Computing the dimensions of sectors needed to make cones of specified dimensions.			
• Explaining the relation between the base radius, height and slant height of a cone.			
• Explaining the methods to compute surface areas and volumes of cones.			
• Explaining the methods to compute the surface areas and volumes of spheres and hemispheres.			

Appendix

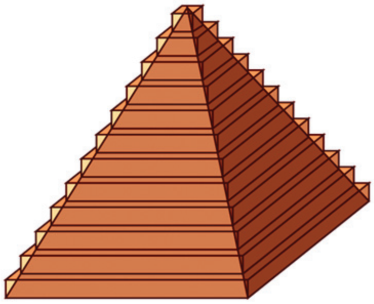
We have seen only the techniques of calculating volumes of pyramids and cones, and also the surface area and volume of a sphere. For those who may be interested in knowing how they are actually got, we give some explanations below.

Volume of a pyramid

We can think of a stack of square plates, of decreasing size as an approximation to a square pyramid.



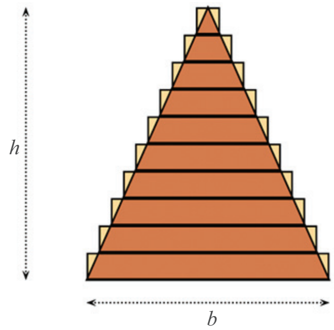
As we decrease the thickness of the plates and increase their number, we get better approximations.



And the sum of the volumes of these plates get nearer to the volume of the pyramid.

Suppose we use 10 plates, to start with. Each plate is a square prism of small height. Let's use plates of same height. So, if we take the height of the pyramid as h ,each plate is of height $\frac{1}{10}h$. How do we compute the base of each plate?

If we imagine the pyramid and the stack of plates sliced vertically down from the vertex, we get a picture like this.



Starting from the top, we have isosceles triangles of increasing size. Their heights increase at the rate of $\frac{1}{10}h$ for each plate.

Since these triangles are all similar (why?) their bases also increase at the same rate. So, if we take the base edge of the bases of the pyramid to be b, the bases of the triangles starting from the top are $\frac{1}{10}b, \frac{2}{10}b, ..., b$.

So, the volumes of the plates are

$$\left(\frac{1}{10}b\right)^2 \times \frac{1}{10}h, \left(\frac{2}{10}b\right)^2 \times \frac{1}{10}h, ..., b^2 \times \frac{1}{10}h$$

And their sum?

$$\frac{1}{10}b^2h\left(\frac{1}{10^2}+\frac{2^2}{10^2}+...+\frac{9^2}{10^2}+\frac{10^2}{10^2}\right) = \frac{1}{1000}b^2h\left(1^2+2^2+3^2+...+10^2\right)$$

We have seen how such sums can be computed in the section, **Sum of squares** of the lesson, **Arithmetic Sequences**.

$$1^2 + 2^2 + 3^2 + ... + 10^2 = \frac{1}{6} \times 10 \times (10 + 1) \times (2 \times 10 + 1)$$

Thus the sum of the volumes

$$\frac{1}{1000}b^2h\times\frac{1}{6}\times10\times11\times21=\frac{1}{6}b^2h\times\frac{10}{10}\times\frac{11}{10}\times\frac{21}{10}=\frac{1}{6}b^2h\times1.1\times2.1$$

Now imagine 100 such plates (we cannot draw it anyway.)

The thickness of a plate becomes $\frac{1}{100}h$ and the base edges would be $\frac{1}{100}b, \frac{2}{100}b, \frac{3}{100}b, ..., b$. So the sum of the volumes would be

$$\frac{1}{100^3}b^2h\left(1^2+2^2+3^2+...+100^2\right) = \frac{1}{100^3}b^2h\times\frac{1}{6}\times100\times101\times201$$

$$=\frac{1}{6}b^2h\times\frac{100}{100}\times\frac{101}{100}\times\frac{210}{100}$$

$$=\frac{1}{6}b^2h\times1.01\times2.01$$

What if we increase the number of plates to 1000? Without going through detailed computations, we can see that the sum of volumes would be

$$\frac{1}{6}b^2h\times1.001\times2.001$$

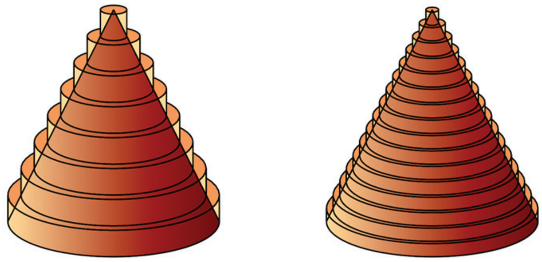
What is the number to which these sums get closer and closer to?

It is the volume of the pyramid; and it is

$$\frac{1}{6}b^2h\times1\times2=\frac{1}{3}b^2h$$

Volume of a cone

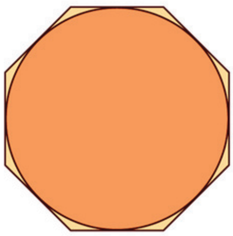
Just as we stacked square plates to approximate a pyramid, we can stack circular plates to approximate a cone.



And in much the same way, we can compute the volume of a cone also. (Try!)

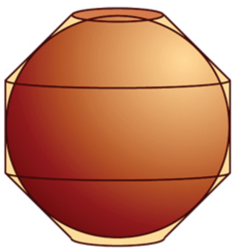
Surface area of a sphere

First consider a circle through the middle of the sphere and a regular polygon with its sides touching it;



Now if this figure revolves, we get the sphere and a solid just covering it.

In the picture above, this solid can be split into two frustums and a cylinder.



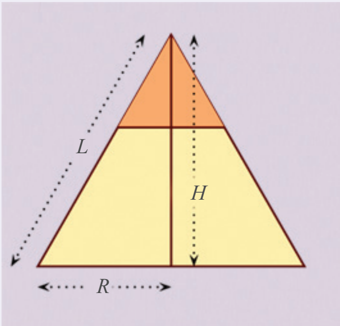
Small and large

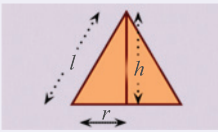
Cutting a cone parallel to the base, we get a small cone on top.

What is the relation between the dimensions of the small and large cones?

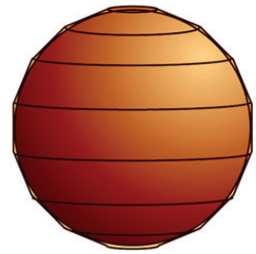
If we take the base radius, height and slant height of the large cone as R, H, L and those of the small cone as r, h, l, we get

$$\frac{r}{R} = \frac{h}{H} = \frac{l}{L}$$



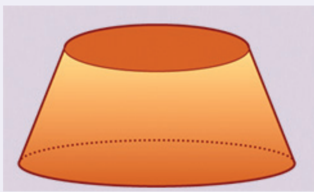


As we increase the number of sides of the polygon, the covering solid approximates the sphere better.

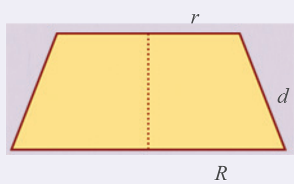


Frustum of a cone

If we cut off a small cone from the top of a cone, the remaining piece is called the frustum of a cone.



How do we find the area of the curved surface of a frustum in terms of its slant height and base radii?



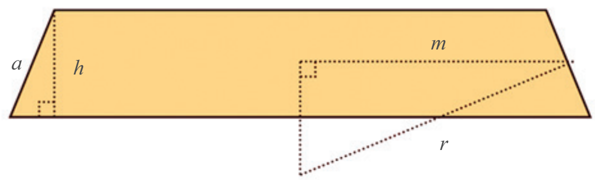
Taking the slant heights of the large and small cones as L and l, we get the d in the figure above as $d = L - l$. So the surface area of the frustum is,

$$\begin{aligned}\pi RL - \pi rl &= \pi(RL - rl)\\ &= \pi(R(l+d) - rl)\\ &= \pi(Rl + Rd - rl)\end{aligned}$$

Now, as seen earlier, we have $\frac{r}{R} = \frac{l}{L}$. So that $Rl = rL$. Using this, the area of the curved surface is,

$$\begin{aligned}\pi(rL + Rd - rl) &= \pi(r(L-l) + Rd)\\ &= \pi(rd + Rd)\\ &= \pi(r+R)d\end{aligned}$$

To compute the area of the curved surface of these frustums, let's consider one of these. Let's take its height as h and the radius of its middle circle as m. Let's also take the radius of the sphere as r and the length of a side of the covering polygon as a. We then have a figure like this.



The two right triangles in the figure are similar and so

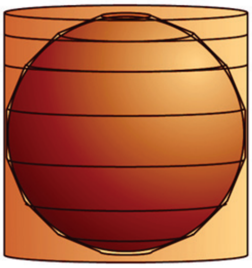
$$\frac{m}{r} = \frac{h}{a}$$

which can be written as

$$am = rh$$

The area of the curved surface of the frustum got by revolving this is $2\pi ma$, as shown in the side bar. **Frustum and cylinder**, on the last page. And this is equal to $2\pi rh$, by the above equation; that is, the area of the curved surface of a cylinder of base radius r and height h.

So what do we get? In the solid which approximates the sphere, the curved surface area of each frustum is equal to that of a cylinder of the same height with base radius equal to that of the sphere.



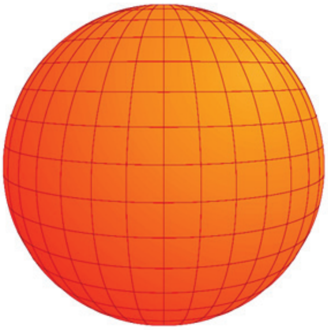
So the curved surface area of the whole approximating solid is equal to the sum of the curved surface areas of all these cylinders. And what do we get on putting together all these cylinders? A large cylinder, just covering the sphere.

As we increase the number of sides of the polygon covering the circle, it becomes more circle-like; and the solid covering the sphere becomes more sphere-like. As seen just now, the curved surface area of any such solid is equal to the curved surface area of a cylinder just covering the sphere. So the surface area of the sphere is also equal to the area of the curved surface of this cylinder. Since the base radius of the cylinder is r and its height is $2r$, the area of its curved surface is

$$2\pi \times r \times 2r = 4\pi r^2$$

Volume of a sphere

See these pictures:

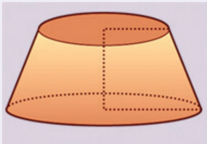


A sphere is divided into cells by horizontal and vertical circles. If we join the corners of such a cell to the centre of the sphere, we get a pyramid-like solid.

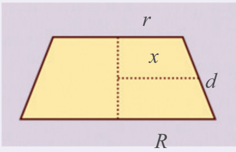
The sphere is made up of such solids joined together; and so the volume of the sphere is the sum of the volumes of these solids. Now if we change each cell into an actual square which touches the sphere, we get a solid which just covers the sphere; and the solid is made up of actual square pyramids. The heights of all these pyramids are equal to the radius of the sphere. If we take it as r and the base area of a pyramid as a, the volume of a pyramid is $\frac{1}{3}ar$.

Frustum and cylinder

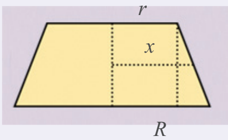
We have seen that the area of the curved surface of a frustum is $\pi(r + R)d$



Taking the radius of the circle round its middle as x, we get a figure like this:



Let's draw one more line:



From the two similar right triangles on the right,

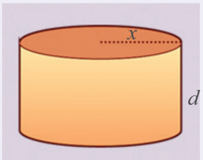
$$\frac{x - r}{R - r} = \frac{1}{2}$$

Simplifying,this gives

$$x = \frac{1}{2}(R + r)$$

So, the area of the curved surface of the frustum can be written $2\pi xd$.

But this is the area of the curved surface of a cylinder of base radius x and height d.



The volume of the solid covering the sphere is the sum of the volumes of all such pyramids. The bases of all such pyramids make up the surface of the solid, so that the sum of the base areas of the pyramids is the surface area of this solid. If it is taken as s, the volume of the solid would be $\frac{1}{3}sr$.

As we decrease the size of the cells and increase their number, the solid covering the sphere approximates the sphere better; and the surface areas of the solid gets closer to the surface area of the sphere. Since the surface area of the sphere is $4\pi r^2$, the volume of the solid get closer to the number

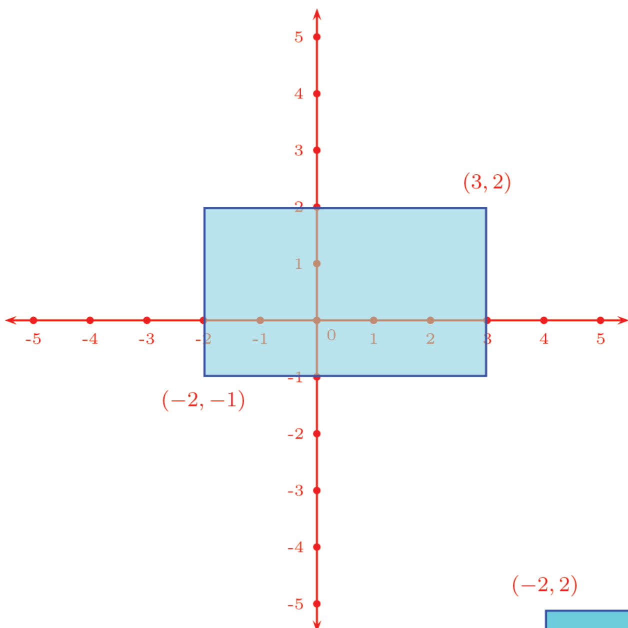
$$\frac{1}{3} \times 4\pi r^2 \times r = \frac{4}{3}\pi r^3$$

And this is the volume of the sphere.

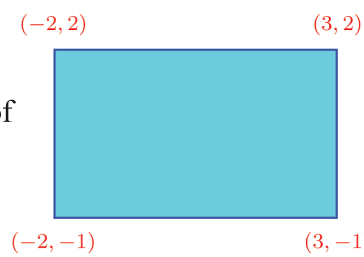
9 Geometry and Algebra

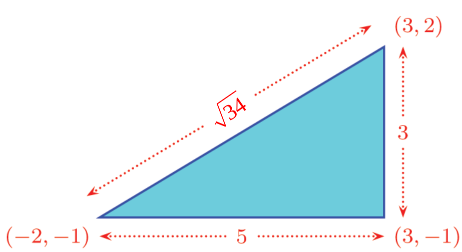
Triangles

We have seen that if the line joining two points is not parallel to either axes, then we can draw a rectangle with these points as opposite vertices and sides parallel to the axes:



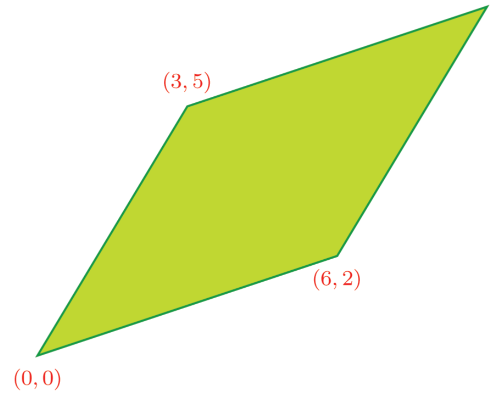
Moreover, we have seen how we can find the coordinates of the other two vertices without drawing to the axes.

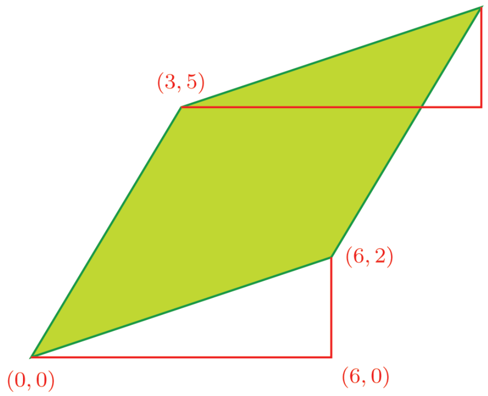




It is using such a rectangle that we computed the distance between two points like these, in terms of their coordinates. In fact, we didn't use the full rectangle, but only a right triangle forming half of it.

Computations using such right triangles are useful in many situations. For example, see this parallelogram with the origin and two other points as vertices.

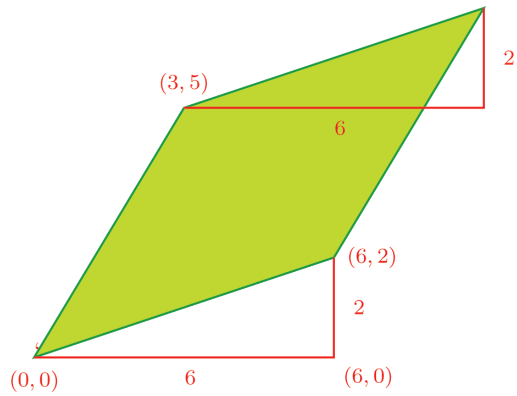


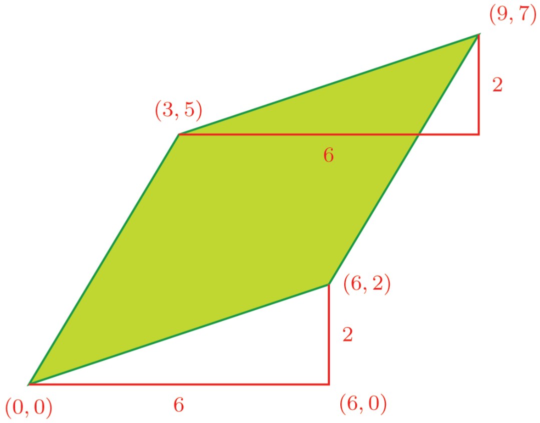


We have to find the fourth vertex.

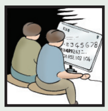
For that, we draw right triangles with the top and bottom sides as hypotenuse and the perpendicular sides parallel to the axes.

In these triangles, the hypotenuse and the angles at its ends are equal. (Why?) So, their perpendicular sides are also equal. We can easily find the lengths of the perpendicular sides of the lower triangle. These are the lengths of the perpendicular sides of the upper triangle also.

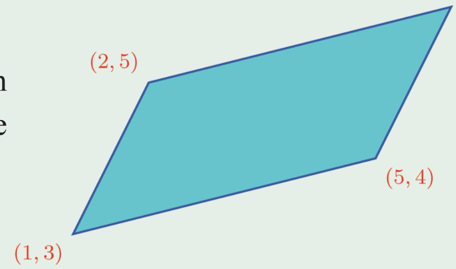




Now we can calculate the lower right corner of the upper triangle as (9, 5) and the top right corner as (9, 7). (How?)

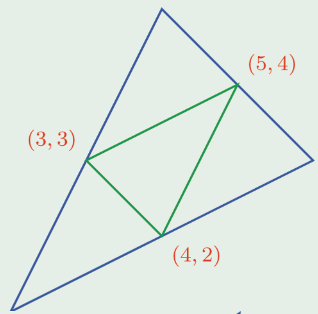


(1) What are the coordinates of the fourth vertex of the parallelogram shown on the right?

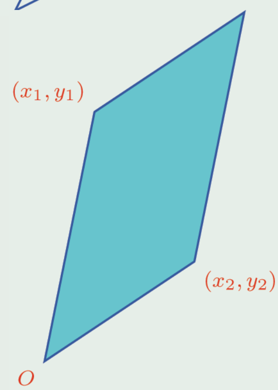


(2) In this picture, the mid points of the sides of the large triangle are joined to make a small triangle inside.

Calculate the coordinates of the vertices of the large triangle.



(3) A parallelogram is drawn with the lines joining (x_1, y_1) and (x_2, y_2) to the origin as adjacent sides. What are the coordinates of the fourth vertex?

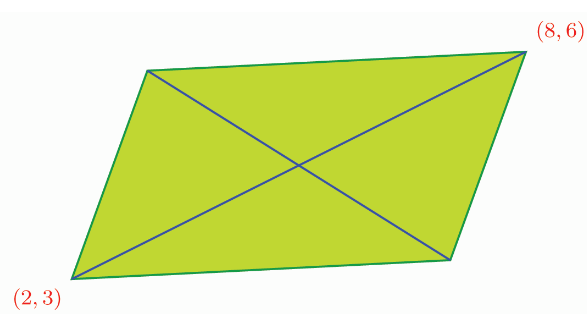


(4) Prove that in any parallelogram, the sum of the squares of all sides is equal to the sum of the squares of the diagonals.

Ratio

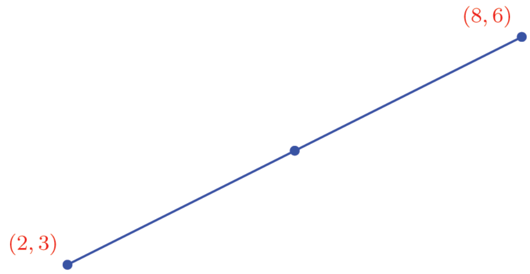
Look at this problem:

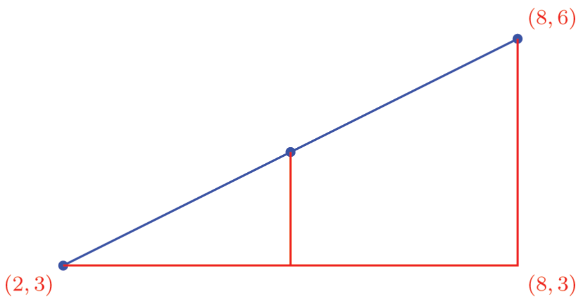
Find the coordinates of the point of intersection of the diagonals of the parallelogram with opposite vertices (2, 3) and (8, 6).



The diagonals of a parallelogram bisect each other, right? So the point of intersection of the diagonals is the mid point of each.

Let's first draw a diagonal:



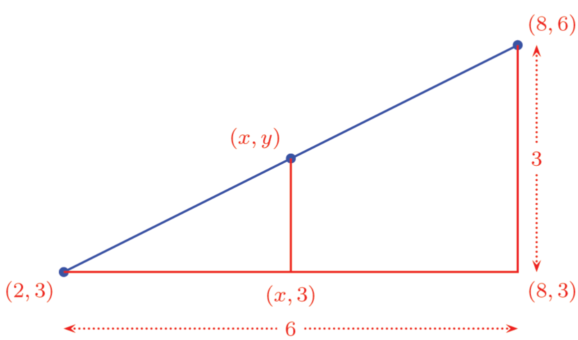


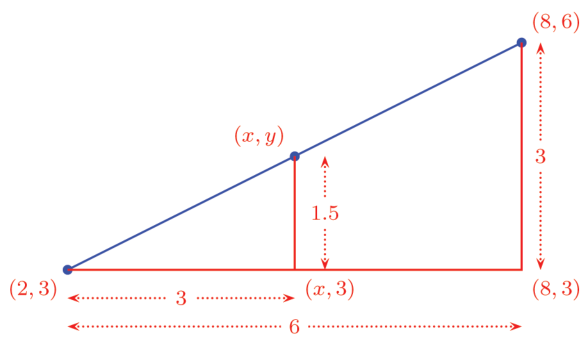
Here also, we draw right triangles with sides parallel to the axes-one with the full diagonal as hypotenuse and another with half of the diagonal as hypotenuse.

The vertical sides of the large and small triangles are parallel. So these right triangles have the same angles. (Why?)

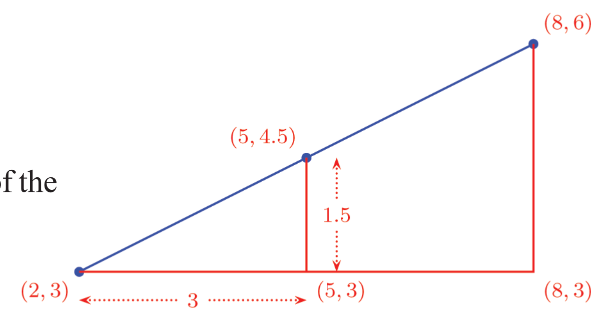
Thus the ratio of the sides are also equal.

Since the hypotenuse of the small triangle is half that of the large triangle, the perpendicular sides of the small triangle are also half of the large triangle. And we know the perpendicular sides of the large triangle. So we can calculate the perpendicular sides of the smaller triangle.



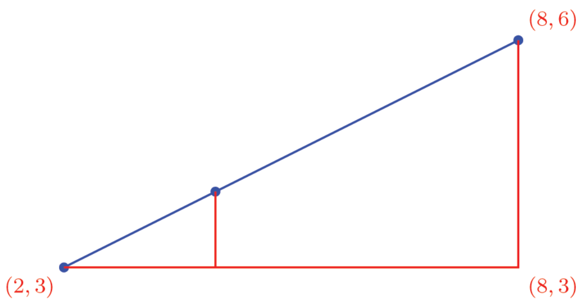


Now, can't we find the coordinates of the mid point we seek?



Like this, find the mid point of the line joining (–3, 5) and (7, 3).

Suppose, instead of the mid point, we have to find the coordinates of the point dividing a line in some other ratio. For example in the problem above, suppose we have to find the coordinates of the point dividing the diagonal in the ratio 1 : 2. We can use the same method, with suitable changes.

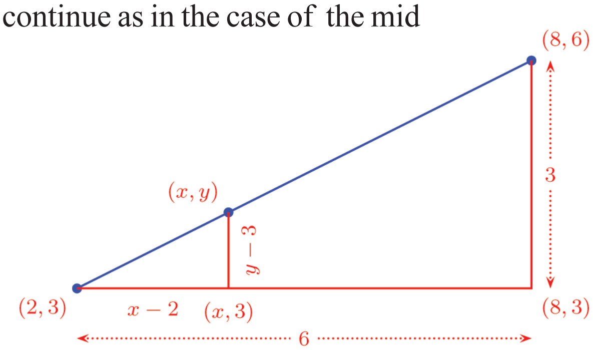


Here, since the ratio of the parts of the line is 1 : 2, the smaller part is $\frac{1}{3}$ of the whole line.

Thus the hypotenuse of the small triangle is $\frac{1}{3}$ of the hypotenuse of the large triangle.

So, the perpendicular sides of the small triangle is also $\frac{1}{3}$ of the perpendicular sides of the large triangle. Now we can continue as in the case of the mid point.

In a slightly different way, we can do it using some algebra. Take the coordinates to be found as (x, y). Then the lengths of the sides of the triangle are like this.



We can write the relations between the perpendicular sides of the triangle like this:

$$\frac{x-2}{6} = \frac{y-3}{3} = \frac{1}{3}$$

From this we get

$$\frac{x-2}{6} = \frac{1}{3}; \quad \frac{y-3}{3} = \frac{1}{3}$$

and so

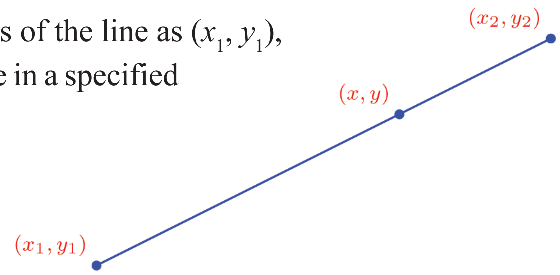
$$x = 6 \times \frac{1}{3} + 2 = 4$$

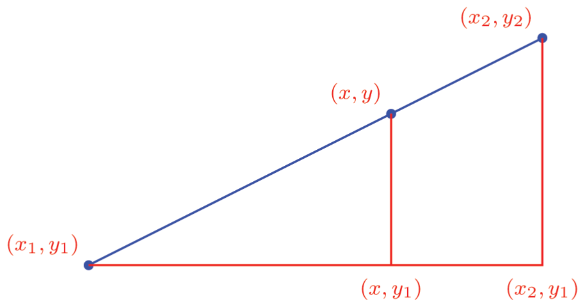
$$y = 3 \times \frac{1}{3} + 3 = 4$$

Thus the coordinates of the point dividing the line in the ratio 1 : 2 is (4, 4).

Let's look at this method in a general setting , so that we can use it without drawing pictures every time.

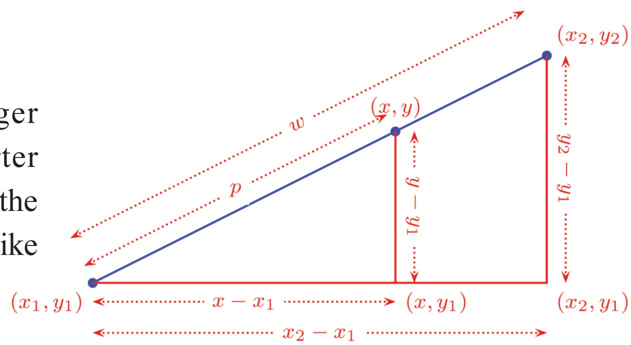
Let's take the coordinates of the ends of the line as (x_1, y_1), (x_2, y_2), and the point dividing the line in a specified ratio as (x, y).





As before, let's draw two right triangles with sides parallel to the axes-one with the whole line as hypotenuse and another with a part of it as hypotenuse.

Taking the length of the longer hypotenuse as w and the shorter hypotenuse as p, we can mark the lengths of all sides of the triangles like this.



The equality of the ratios of sides can then be written like this

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{p}{w}$$

Here, w is the length of the whole line and p is the length of a part of it. If we know the ratio of the parts, we can also compute the ratio of a part and the whole. So, we know $\frac{p}{w}$. Now from the equations above, we can compute x and y.

$$x = x_1 + \frac{p}{w}(x_2 - x_1)$$

$$y = y_1 + \frac{p}{w}(y_2 - y_1)$$

For example, let's calculate the coordinates of the point which divides the line joining (2, 4) and (8, 7) in the ratio 3 : 5. Taking this point as (x, y), the part of the line from (2, 4) to (x, y) is $\frac{3}{8}$ of the whole line. So, from our general equations,

$$x = 2 + \frac{3}{8} \times (8 - 2) = 4\frac{1}{4}$$

$$y = 4 + \frac{3}{8} \times (7 - 4) = 5\frac{1}{8}$$

Thus the point we seek is $\left(4\frac{1}{4},\ 5\frac{1}{8}\right)$

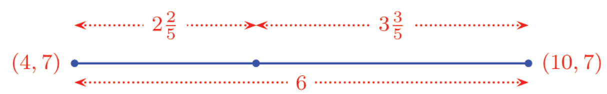
If the line joining two points is parallel to any axis, we cannot draw triangles like this. But in such cases, we can easily find the coordinates of a point dividing the line in a specified ratio.

For example, the line joining the points (4, 7) and (10, 7) is parallel to the x axis (at a distance 7 from the x axis). What are the coordinates of the point dividing this line in the ratio 2 : 3?

The length of this line is $10 - 4 = 6$, right? So, we can calculate the lengths of the parts divided in the ratio 2 : 3.

$$6 \times \frac{2}{5} = 2\frac{2}{5}$$

$$6 \times \frac{3}{5} = 3\frac{3}{5}$$



Thus we find that the coordinates of the point dividing the line in the ratio as $\left(6\frac{2}{5}, 7\right)$.

We often have to find the mid point of the line joining two points. In general, what are the coordinates of the mid point of the line joining (x_1, y_1) and (x_2, y_2)?

In this case, both parts are half the whole line.

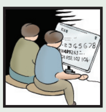
So, in the general equation, we take $\frac{p}{w}$ as $\frac{1}{2}$. Thus in this case,

$$x = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$$

$$y = y_1 + \frac{1}{2}(y_2 - y_1) = \frac{1}{2}(y_1 + y_2)$$

The midpoint of the line joining (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{1}{2}(x_1 + x_2),\ \frac{1}{2}(y_1 + y_2)\right)$$



(1) The coordinates of two points A, B are (3, 2) and (8, 7).

i) Calculate the coordinates of the point P on AB such that $AP : PB = 2 : 3$

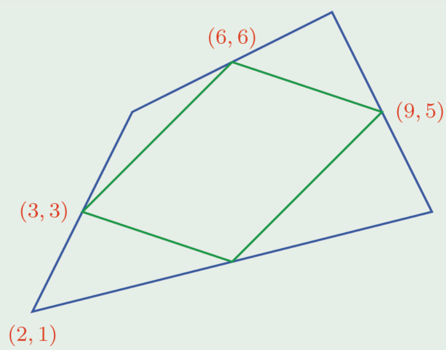
ii) Calculate the coordinates of the point Q on AB such that $AQ : QB = 3 : 2$

(2) The coordinates of the vertices of a quadrilateral are (2, 1), (5, 3), (8, 7), (4, 9) in order.

i) Find the coordinates of the midpoints of all sides.

ii) Prove that the quadrilateral with these mid points as vertices is a parallelogram.

(3) In the picture, the mid points of the sides of the large quadrilateral are joined to draw the small quadrilateral inside.



i) Find the coordinates of the fourth vertex of the small quadrilateral.

ii) Find the coordinates of the other three vertices of the large quadrilateral.

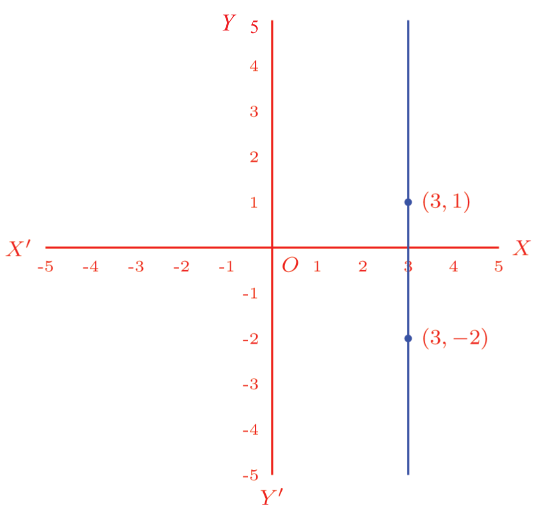
(4) The vertices of a triangle are the points with coordinates (3, 5), (9, 13), (10, 6). Prove that the triangle is isosceles. Calculate its area.

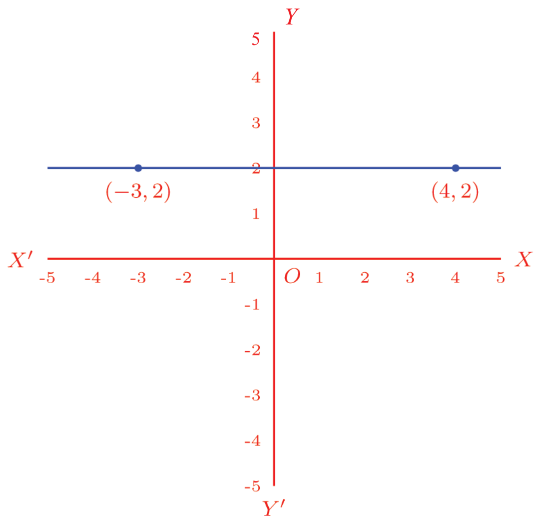
(5) The coordinates of the vertices of a triangle are (–1, 5), (3, 7), (3, 1). Find the coordinates of its centroid.

(6) The centre of a circle is (1, 2) and (3, 2) is a point on it. Find the coordinates of the other end of the diameter through this point.

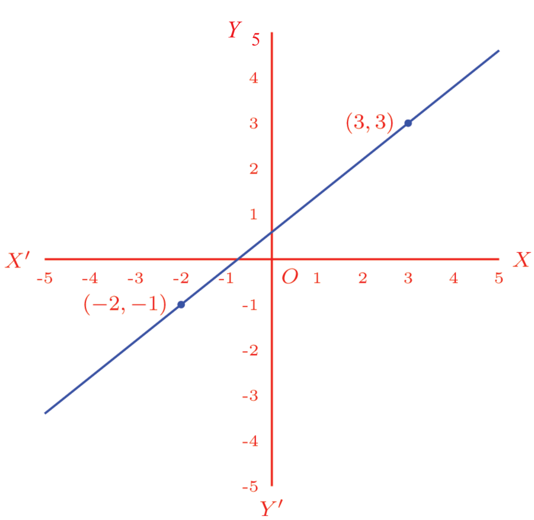
Straight lines

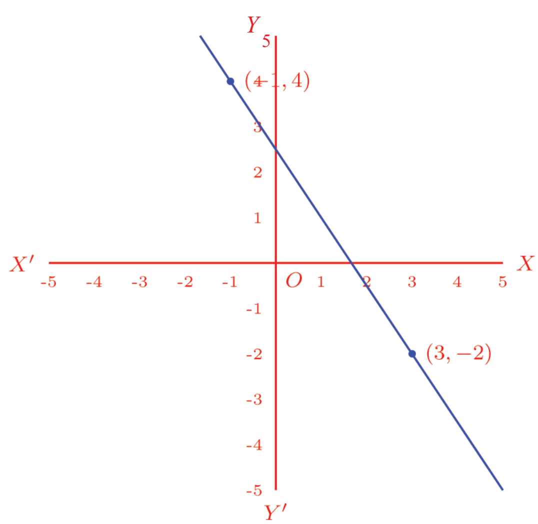
We can draw a line (and only one line) joining any two points; and we can extend it as much as we want to either side. If the x coordinates of the first two points are the same, the line would be parallel to the y axis; and if the y coordinates are equal, the line would be parallel to the x axis.



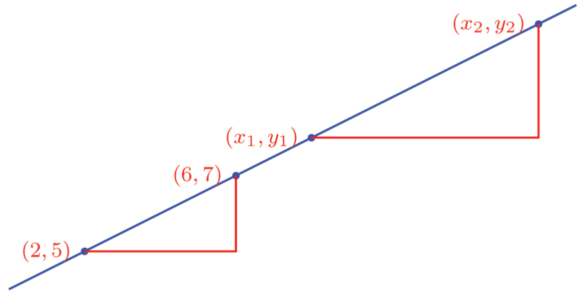


If both x and y coordinates are different, the line will be slanted, not parallel to either axes.

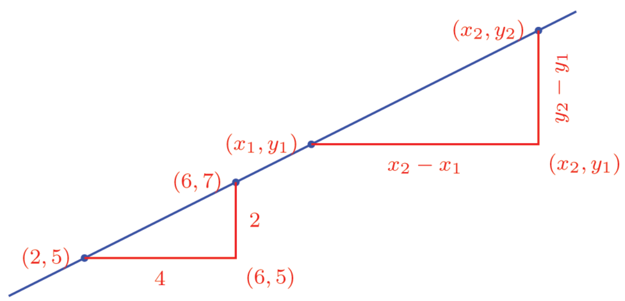




Moving along this line, the x and y coordinates change at every point. There is a connection between these changes. See this picture.



(x_1, y_1) and (x_2, y_2) are two points on the line joining (2, 5) and (6, 7). We draw two right triangles with these parts of the line as hypotenuse and the perpendicular sides parallel to the axes. The sides of these triangles are in the same ratio, right?



So,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2}{4} = \frac{1}{2}$$

Thus we can write this as

$$y_2 - y_1 = \frac{1}{2}(x_2 - x_1)$$

Here, (x_1, y_1) and (x_2, y_2) can be any two points on the line.

> For any two points on the line joining (2, 5) and (6,7), the difference in y coordinates is half the difference in x coordinates.

We can put this in another manner. As we move from one point to another along this line, both the x and y coordinates change and the change is like this:

Moving along the line joining (2, 5) and (6, 7), the change in y is half the change in x at every stage.

What if we take another pair of points instead of (2, 5) and (6, 7)?

For example, let's take (1, 4) and (5, 12). As we move from (1, 4) to (5, 12) along the line joining these two points, the x coordinate increases by 4; and the y coordinate increases by 8. Thus the change in y is twice the change in x. This happens at any two positions on this line.

Moving along the line joining (1, 4) and (5, 12), the change in y is twice the change in x at every stage.

In both these lines as x increases, so does y. It can be otherwise.

For example, taking the points (3, 6) and (7, 4), as the x coordinate increases by 4, the y coordinate decreases by 2. Thus.

Moving along the line joining(3, 6) and (7, 4), the change in y is the negative of half the change in x.

What do we see in general in all these?

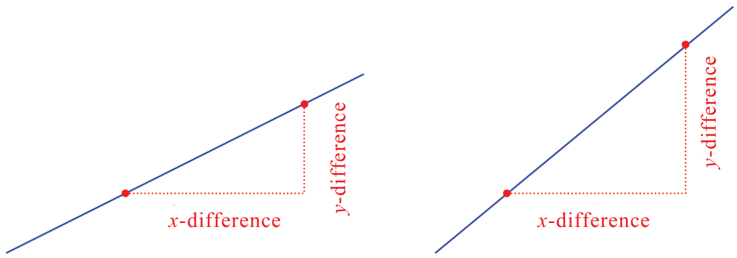
In any line not parallel to either axis, the change in y coordinate is equal to the product of the change in x coordinate and a fixed number.

We have noted that such changes have a special name:

In any line not parallel to either axis, the change in y is proportional to the change in x.

In a line parallel to the x axis, the y coordinate does not change. So the change in y for two points on this line is 0. This is the x difference multiplied by 0. So in this case also, we can say that the y difference is the product of the x difference and a fixed number. But the x, y change is not proportional.

Geometrically, x difference is horizontal shift, while y difference is vertical shift. So what we get on dividing the y difference by the x difference is the rate of vertical shift with respect to the horizontal shift.



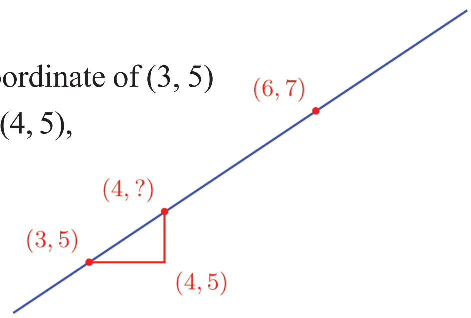
In other words, the constant of proportionality of the change in coordinates is a measure of the slant of the line. It is called the *slope* of the line.

We can use this idea to find other points on a line joining two specified points.

For example, let's take the line joining (3, 5) and (6, 7). For these points, the change in x is 3 and the change in y is 2.

So, anywhere on this line, for a change of 3 in x, the change in y is 2; that is, as the x coordinate changes by 1 the y coordinate changes by $\frac{2}{3}$, every where on this line.

Now suppose we increase the x coordinate of (3, 5) by 1 to make it 4. The line through (4, 5), parallel to the y axis meets this line at some point.



To find the slope of a line drawn in GeoGebra, select Slope and click on the line. In the number so got, the x difference is taken as 1. So slope is the y difference.

What is the y coordinate of the point? Its x coordinate is 1 more than 3. So, to get the y coordinate, we must add $\frac{2}{3}$ to 5.

Thus $\left(4,\ 5\frac{2}{3}\right)$ is the point on this line.

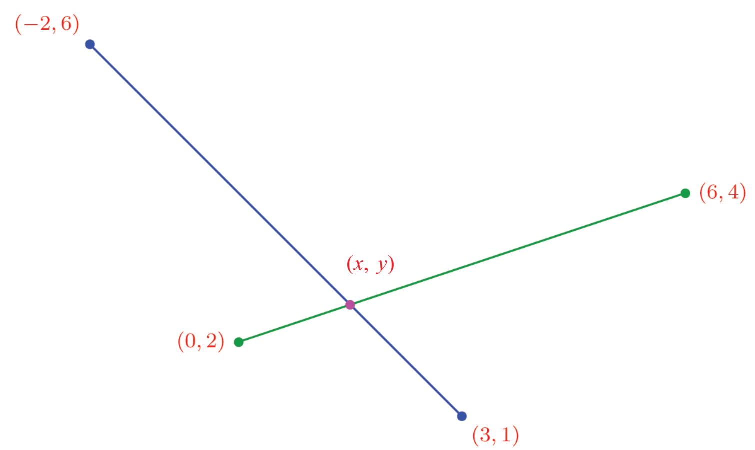
In the same way we can find a point on this line, with x coordinate any number.

For example, what is the point on this line with x coordinate 9?

9 is 6 added to 3; so to get the y coordinate, we must add $6 \times \frac{2}{3} = 4$ to 5. Thus (9, 9) is a point on this line.

We saw that the points (3, 5) , (6, 7), (9, 9) are points on the same line. Any relation between the x coordinates 3, 6 and 9 of these points? What about the y coordinates 5, 7, 9? Can you find other points on this line with natural number coordinates?

We can use this idea to find the point of intersection of two lines also. For example, let (x, y) be the point of intersection of the line joining (0, 2), (6, 4) and the line joining (3, 1), (–2, 6).



So (x, y) is on both lines. Since the x, y changes are proportional in any line, we get from the first line

$$\frac{y-2}{x-0} = \frac{4-2}{6-0}$$

And from the second line

$$\frac{y-1}{x-3} = \frac{6-1}{-2-3}$$

These equations can be simplified like this:

$$x - 3y = -6$$
$$x + y = 4$$

We have seen in Class 9, how numbers satisfying two equations like this can be found out. Using it, we get.

$$x = 1\frac{1}{2} \qquad y = 2\frac{1}{2}$$

Thus the lines intersect at $\left(1\frac{1}{2}, 2\frac{1}{2}\right)$.

(1) Prove that the points (1, 8), (2, 5), (3, 7) are on the same line.

(2) Find the coordinates of two other points on the line joining (–1, 4) and (1, 2).

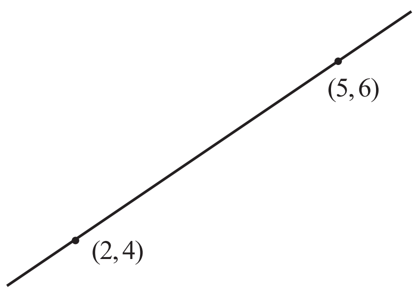
(3) $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are arithmetic sequences. Prove that all points with coordinates in the sequence $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ are on the same line.

(4) Prove that if the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are on a line, so are $(3x_1 + 2y_1, 3x_1 - 2y_1)$, $(3x_2 + 2y_2, 3x_2 - 2y_2)$, $(3x_3 + 2y_3, 3x_3 - 2y_3)$. Would this be true if we take other numbers instead of 3 and 2?

Figures and equations

If we take (x, y) as a point on the line joining (2, 4) and (5, 2), we get

$$\frac{y-4}{x-2} = \frac{6-4}{5-2} = \frac{2}{3}$$



Consider the points (1,3), (5,6) and the points (2,4), (6,7). In both pairs of points, the y difference divided by the x difference is $\frac{3}{4}$. Draw the line joining each pair in GeoGebra. What is the relation between these lines?

We can write this equation as

$$3(y - 4) = 2(x - 2)$$

Simplifying further, we get

$$2x - 3y + 8 = 0$$

What does this mean?

If we take any point on this line, its coordinates satisfy this equation; that is, if the point with coordinates (p, q) is on this line, then $2p - 3q + 8 = 0$.

On the other hand, if we take a pair of numbers satisfying this equation, would the point with coordinates these points be on this line?

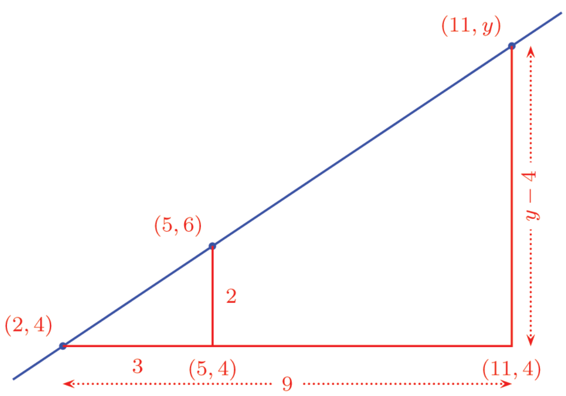
For example,

$$x = 11, y = 10 \text{ gives}$$

$$2x - 3y + 8 = 22 - 30 + 8 = 0$$

Is the point (11, 10) on this line?

As seen before, the line through (11, 4) parallel to the y axis meets this line at some point. The x coordinate of this point is 11 itself. Let's take the y coordinate as y.



From the figure, we see that

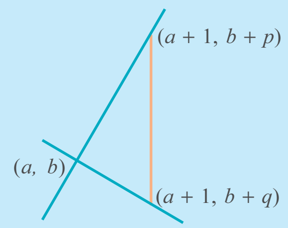
$$\frac{y-4}{11-2} = \frac{2}{3}$$

Simplifying this, we get $y = 10$. So the point is on this line.

Slope and perpendicular

It is not difficult to see that parallel lines have the same slope. What is the relation between the slopes of lines perpendicular to each other?

Consider two lines of slopes p and q, intersecting at a point (a, b). Then $(a + 1, b + p)$ is a point on the first line and $(a+1, b+q)$ is a point on the second. (Why?)



If the lines are mutually perpendicular, then $(a, b), (a + 1, b + p), (a + 1, b + q)$ are the vertices of a right triangle and the hypotenuse is the line joining the second and third points. The squares of the lengths of the perpendicular sides of this triangle are $p^2 + 1, q^2 + 1$ and the hypotenuse is $|p - q|$. So

$$(p^2 + 1) + (q^2 + 1) = (p - q)^2$$

Simplifying this, we get

$$2 = -2pq$$

That is,

$$pq = -1$$

Thus in two lines perpendicular to each other, the slope of one is the negative of the reciprocal of the other.

Now suppose we have found a pair of numbers p, q such that $2p - 3q + 8 = 0$. The line through $(p, 4)$, parallel to the y axis, meets

If we type $2x - 3y + 8 = 0$ in the Input Bar of GeoGebra, a line satisfying the equation is obtained. Make three slider a, b, c and type $ax + by + c = 0$ in the Input Bar. Observe the change in the line as the change in the slider values.

the line joining (2, 4), (5, 6) at some point with x coordinate p. Taking this point as (p, y), we get as in the example,

$$\frac{y-4}{p-2} = \frac{2}{3}$$

Simplification of this gives

$$y = \frac{2}{3}(p-2)+4$$

Now from the equation $2p - 3q + 8 = 0$, we get

$$q = \frac{2}{3}(p-2)+4$$

So $y = q$, and thus (p, q) is a point on this line.

What do we see here?

> The set of coordinates of points on the line joining (2, 4), (5, 6), and the set of number pairs satisfying the equation $2x - 3y + 8 = 0$ are the same.

We can shorten this statement like this:

> The equation of the line joining (2,4) and (5, 6) is $2x - 3y + 8 = 0$.

Similarly, once we have the coordinates of any two points on a line, we can write its equation.

Let's find the equation of the line joining the points (0, 0) and (1, 1). Taking the coordinates of a point on this line as (x, y), we get,

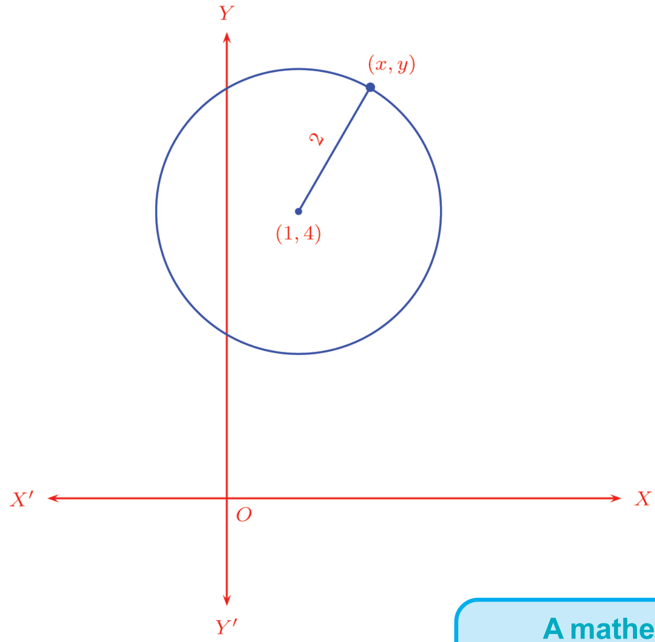
$$\frac{y-0}{x-0} = \frac{1-0}{1-0}$$

Simplifying this, we get $y = x$. This is the equation of this line.

From this, we see that for any point on this line, the x and y coordinates are the same.

We can find equations for not only straight lines, but for other geometric figures also. For example, let's look at the circle with centre (1, 4) and radius 2.

The distance of any point on this circle to the centre is 2.



We have seen that the square of the distance is $(x-1)^2 + (y-4)^2$. Since this is equal to the square of the radius,

$$(x-1)^2 + (y-4)^2 = 4$$

The coordinates of any point on the circle satisfies this equation; on the other hand any pair of numbers satisfying this equation form the coordinates of a point on this circle.

So, this is the equation of the circle. We can also expand this and simplify to write it as

$$x^2 + y^2 - 2x - 8y + 13 = 0$$

So, what is the equation of the circle centred at the origin and of radius 1?

A mathematical union

Descartes gave a method to transform geometric figures to equations and vice versa, by representing points as pairs of numbers. This branch of mathematics, unifying algebra and geometry (which developed along different ways till then) is called Analytic Geometry.

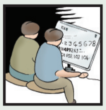
This new vision is the basis of another branch of mathematics, known as Calculus, which produced great changes in mathematics and other sciences.

Progress always occurs through the synthesis of dualities.

Taking the coordinates of a point on this circle as (x, y), we get the distance from the centre as $x^2 + y^2$; since it is the square of the radius, we get

$$x^2 + y^2 = 1$$

This is the equation of the circle.



(1) Find the equation of the line joining (1, 2) and (2, 4). In this, find the sequence of y coordinates of those points with the consecutive natural numbers 3, 4, 5, ... as the x coordinates.

(2) Find the equation of the line joining (–1, 3), (2, 5). Prove that if (x, y) is a point on this line, so is $(x + 3, \ y + 2)$.

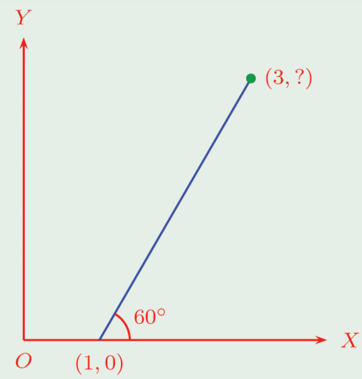
(3) Prove that for any number x, the point $(x, 2x + 3)$ is on the line joining (–1, 1), (2, 7).

(4) The x coordinate of a point on the slanted (blue) line in the picture is 3.

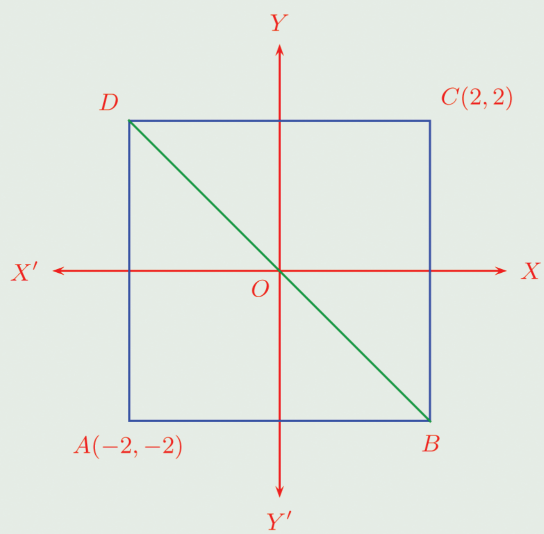
i) What is its y coordinate?

ii) What is the slope of the line?

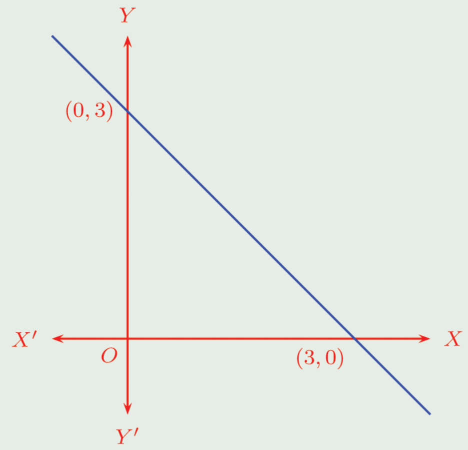
iii) Write the equation of the line.



(5) In the picture, $ABCD$ is a square. Prove that for any point on the diagonal BD, the sum of the x and y coordinates is zero.



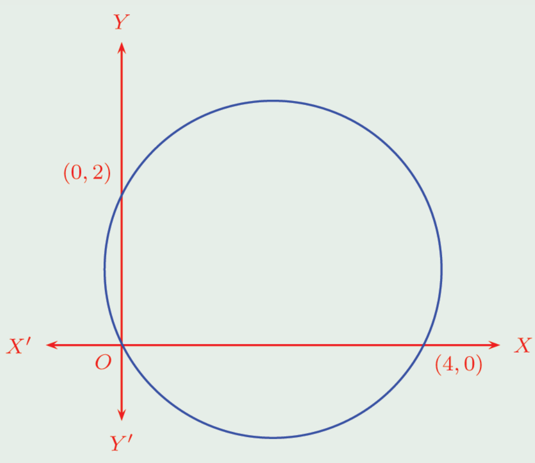
(6) Prove that for any point on the line intersecting the axes in this picture, the sum of the x and y coordinates is 3.



(7) Find the equation of the circle with centre at the orgin and radius 5. Write the coordinates of eight points on this circle.

(8) Let (x, y) be a point on the circle with the line joining (0, 1) and (2, 3) as diameter. Prove that $x^2 + y^2 - 2x - 4y + 3 = 0$. Find the coordinates of the points where this circle cuts the x axis.

(9) What is the equation of the circle in the picture below?



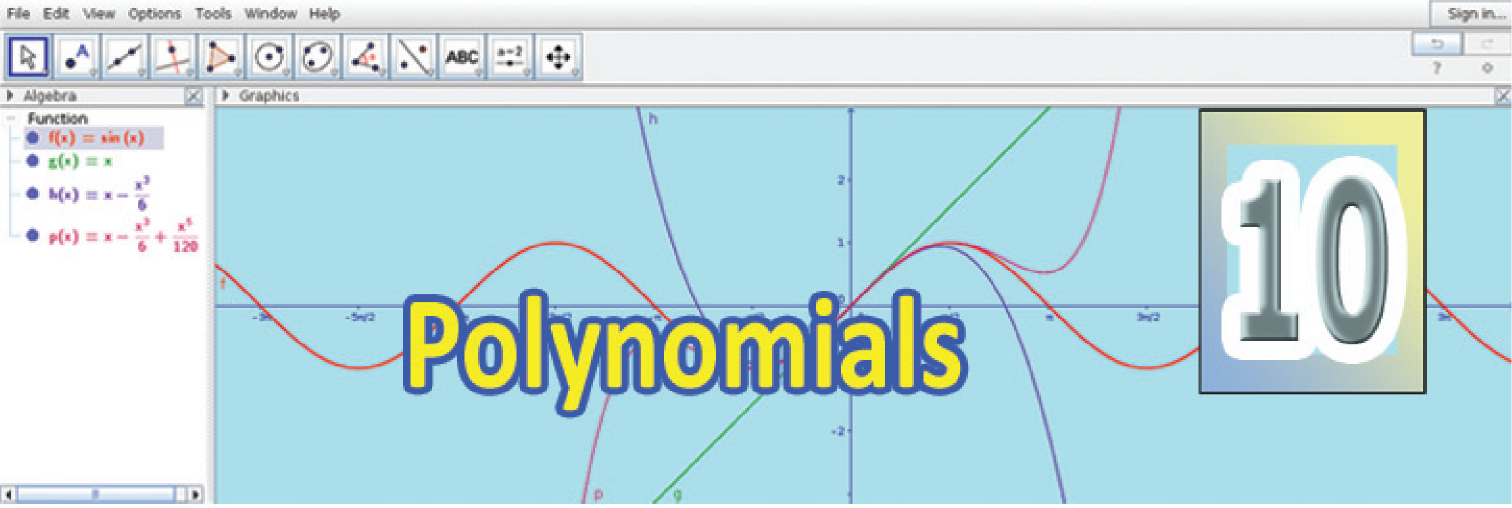
What are the coordinates of the centroid of the triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3)?

If we type any equation connecting x, y in the Input Bar of GeoGebra, the figure made up of points whose coordinates satisfy that equation. Type each of the equations below and see the figures.

- `2x^2 + 2y^2 = 4`
- `2x^2 + 3y^2 = 4`
- `2x^2 − 3y^2 = 4`
- `2x^2 + 3y = 4`

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Proving that for any two points on a line, the change in y coordinates is proportional to the change in x coordinates. • Forming the equations of a line joining two points. • Forming the equation of a circle with specified centre and radius			



Factors and solutions

We have seen in Class 8 that the difference of two squares is the product of their sum and difference.

In the language of algebra,

$$x^2 - y^2 = (x + y)\,(x - y) \text{ for all numbers } x, y.$$

Let's take different numbers as y in this:

For any number x,

$$x^2 - 1 = (x + 1)\,(x - 1)$$

$$x^2 - 2 = \left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$$

$$x^2 - \frac{1}{4} = \left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$$

$x^2 - 1$, $x^2 - 2$, $x^2 - \frac{1}{4}$ are all second degree polynomials, while $x - 1$, $x + 1$, $x - \sqrt{2}$, $x + \sqrt{2}$, $x - \frac{1}{2}$, $x + \frac{1}{2}$ are all first degree polynomials.

Thus in all the equations above, a second degree polynomial is written as the product of two first degree polynomials.

When we write a number as the product of two numbers, those numbers multiplied are called factors. For example, since $12 = 2 \times 6$, we call 2 and 6, factors of 12. Similarly, since $x^2 - 1 = (x - 1)\,(x + 1)$, we call $x - 1$ and $x + 1$ factors of $x^2 - 1$.

Thus we make this general definition:

If the polynomial $p(x)$ is the product of the polynomials $q(x)$ and $r(x)$, then $q(x)$ and $r(x)$ are said to be factors of $p(x)$.

Let's look at some more examples: We have

$$(x-1)(x-2) = x^2 - 2x - x + 2 = x^2 - 3x + 2$$

That is,

$$x^2 - 3x + 2 = (x-1)(x-2)$$

Thus the first degree polynomials $x-1$ and $x-2$ are factors of the second degree polynomial $x^2 - 3x + 2$.

We note another thing here:

If we write $p(x) = x^2 - 3x + 2$, what is $p(1)$?

As seen just now, we can split $p(x)$ as the product of its first degree factors:

$$p(x) = (x-1)(x-2)$$

From this, we see that

$$p(1) = (1-1) \times (1-2) = 0 \times (-1) = 0$$

Similarly, we can see that $p(2) = 0$ also.

Thus 1 and 2 are the numbers we must take as x to make $p(x) = 0$. In other words, 1 and 2 are solutions of the polynomial equation $p(x) = 0$ (that is, the equation $x^2 - 3x + 2 = 0$).

Do we get $p(x) = 0$ for any other numbers as x?

If $(x-1)(x-2)$ is to be 0, one of $x-1$ or $x-2$ must be 0, right?

Let's take another example. We can compute the product

$(x-1)(x-2)(x-3) = (x^2 - 3x + 2)(x-3) = x^3 - 6x^2 + 11x - 6$

Putting this in reverse, we can say that $x-1, x-2, x-3$ are the first degree factors of the third degree polynomial, $x^3 - 6x^2 + 11x - 6$.

Here also, by writing

$$p(x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

We can see, as in the first example, that

$$p(1) = 0,\ p(2) = 0,\ p(3) = 0$$

So, here also, 1, 2, 3 are the solutions of the polynomial equation $p(x) = 0$; that is, the equation,

$$x^3 - 6x^2 + 11x - 6 = 0$$

What general principle do we get from these examples?

If the first degree polynomial $x - a$ is a factor of the polynomial, $p(x)$, then $p(a) = 0$.

We can state this in a little more detail:

If the polynomial $p(x)$ can be split into

$$p(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

as a product of first degree polynomials, then $a_1, a_2, \dots, a_n$ are the solutions of the equation $p(x) = 0$.

So one way to solve a polynomial equation is to split the polynomial into a product of first degree factors.

For example, see this equation:

$$x^2 - 5x + 6 = 0$$

We cannot write $x^2 - 5x + 6$ as a product of more than two first degree polynomials (the product of more than two first degree polynomials is a polynomial of degree greater than 2, isn't it?)

So, let's write

$$x^2 - 5x + 6 = (x - a)(x - b)$$

Expanding the product, we get

$$x^2 - 5x + 6 = x^2 - (a + b)x + ab$$

The coefficients of the polynomials on either side of the equation must be the same. For that we must have,

$$a + b = 5$$
$$ab = 6$$

In other words, we must find two numbers with sum 5 and product 6.

A little bit of thinking gives

$$a = 2 \qquad b = 3$$

So we can split $x^2 - 5x + 6$ like this:

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

From this, we see that the solutions of $x^2 - 5x + 6$ are 2 and 3.

Let's look at another equation:

$$x^2 + 2x - 15 = 0$$

Writing

$$x^2 + 2x - 15 = (x - a)(x - b) = x^2 - (a + b)x + ab$$

as in the first problem, we get

$$a + b = -2$$

$$ab = -15$$

3 and 5 are factors of 15. Since the product is negative, one factor must be negative. -3 and 5 don't give the right sum; 3 and -5 do. So,

$$x^2 + 2x - 15 = (x - 3)(x - (-5)) = (x - 3)(x + 5)$$

Thus the solutions of $x^2 + 2x - 15 = 0$ are 3 and -5.

One more example:

$$x^2 - x - 1 = 0$$

To solve this, we write as usual,

$$x^2 - x - 1 = (x - a)(x - b) = x^2 - (a + b)x + ab$$

which gives

$$a + b = 1$$

$$ab = -1$$

How is this possible?

a and b need not be natural numbers. So, we cannot proceed by trial and error as before. Let's do this in a different way.

Recall another identity from Class 8:

$$(a+b)^2-(a-b)^2 = 4ab$$

We can write this as

$$(a-b)^2 = (a+b)^2-4ab$$

In our problem, $a+b=1$ and $ab=-1$, so that

$$(a-b)^2 = 1^2-4\times(-1) = 1+4 = 5$$

From this, we get $a-b=\pm\sqrt{5}$

Taking $a-b=\sqrt{5}$, the sum and difference of a and b are

$$a+b = 1$$

$$a-b = \sqrt{5}$$

We know how to find the numbers from their sum and difference:

$$a=\frac{1}{2}(1+\sqrt{5}) \quad b=\frac{1}{2}(1-\sqrt{5})$$

What if we take $a-b=-\sqrt{5}$?

Then we get

$$a=\frac{1}{2}\left(1-\sqrt{5}\right) \quad b=\frac{1}{2}\left(1+\sqrt{5}\right)$$

Anyhow, we get,

$$x^2-x-1=\left(x-\frac{1}{2}\left(1+\sqrt{5}\right)\right)\left(x-\frac{1}{2}\left(1-\sqrt{5}\right)\right)$$

Thus we see that the solutions of the equation $x^2-x-1=0$ are $\frac{1}{2}(1+\sqrt{5})$ and $\frac{1}{2}(1-\sqrt{5})$.

Now look at this problem:

How do we write $2x^2-7x+6$ as the product of two first degree polynomials?

First we write the polynomial as

$$2x^2-7x+6 = 2\left(x^2-\frac{7}{2}x+3\right)$$

Next we split $x^2 - \frac{7}{2}x + 3$ into the product of two first degree factors as before and write,

$$x^2 - \frac{7}{2}x + 3 = (x - a)(x - b)$$
$$= x^2 - (a + b)x + ab$$

we get

$$a + b = \frac{7}{2}$$
$$ab = 3$$

and from $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a - b)^2 = \left(\frac{7}{2}\right)^2 - 4 \times 3$$
$$= \frac{49}{4} - 12 = \frac{1}{4}$$
$$a - b = \pm\frac{1}{2}$$

Taking $a - b = \frac{1}{2}$, we have

$$a + b = \frac{7}{2}; \quad a - b = \frac{1}{2}$$

which gives $a = 2,\ b = \frac{3}{2}$

Taking $a - b = -\frac{1}{2}$ gives $a = \frac{3}{2}$ and $b = 2$

Anyway, we can write

$$x^2 - \frac{7}{2}x + 3 = \left(x - \frac{3}{2}\right)(x - 2)$$

So we have $2x^2 - 7x + 6 = 2\left(x - \frac{3}{2}\right)(x - 2)$

$$= (2x - 3)(x - 2)$$

Not all second degree polynomials can be split into the product of first degree factors. For example, consider the polynomial $x^2 + 1$. If it is the product of the first degree polynomials $x - a$ and $x - b$, then we have,

$$x^2 + 1 = (x - a)(x - b) = x^2 - (a + b)x + ab$$

which gives

$$a + b = 0$$

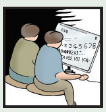
$$ab = 1$$

Let's try to find a, b as before

$$(a - b)^2 = (a + b)^2 - 4ab = 0 - 4 = -4$$

The square of no number is negative. So there are no numbers satisfying the above pair of equations.

Thus $x^2 + 1$ cannot be split as a product of first degree polynomials.



(1) Write each polynomial below as a product of first degree polynomials. Write also the solutions of the equation $p(x) = 0$ in each.

i) $p(x) = x^2 - 7x + 12$ ii) $p(x) = x^2 + 7x + 12$

iii) $p(x) = x^2 - 8x + 12$ iv) $p(x) = x^2 + 13x + 12$

v) $p(x) = x^2 - 2x + 1$ vi) $p(x) = x^2 + x - 1$

vii) $p(x) = 2x^2 - 5x + 2$ viii) $p(x) = 6x^2 - 7x + 2$

(2) Find a second degree polynomial $p(x)$ such that $p(1) = 0$ and $p(-2) = 0$.

(3) Find a second degree polynomial $p(x)$ such that $p(1+\sqrt{3}) = 0$ and $p(1-\sqrt{3}) = 0$.

(4) Find a third degree polynomial $p(x)$ such that $p(1) = 0$, $p(\sqrt{2}) = 0$ and $p(-\sqrt{2}) = 0$.

(5) Prove that the polynomial $x^2 + x + 1$ cannot be written as a product of first degree polynomials.

Polynomial remainder

We have seen that if the polynomial $x - a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$.

Now suppose that for some polynomial $p(x)$ and a number a, we calculate $p(a)$ and find it to be not zero. Can we say that $x - a$ is not a factor of $p(x)$?

If $x - a$ is a factor then $p(a)$ must be zero. But here $p(a)$ is not zero; so $x - a$ is not a factor either.

So our result on first degree factors can be put like this also:

For a polynomial $p(x)$ and a number a, if $p(a) \neq 0$ then $x - a$ is not a factor of $p(x)$.

For example, if we take $p(x) = x^2 - 3x + 3$ and $a = 1$, then we get $p(a) = 1$. So $x - 1$ is not a factor of $x^2 - 3x + 3$.

But we have seen that,

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

So, we can write

$$x^2 - 3x + 3 = (x - 1)(x - 2) + 1$$

This is similar to writing a natural number as the sum of a multiple of another and a remainder, when the latter is not a factor. 6 is not a factor of 15 but we can write

$$15 = (6 \times 2) + 3$$

We use the terms *quotient* and *remainder* for polynomials also, as in the case of natural numbers. Thus in $x^2 - 3x + 3 = (x - 1)(x - 2) + 1$, the polynomial $(x - 2)$ is called the quotient on dividing $x^2 - 3x + 3$ by $x - 1$ and 1 is called the remainder.

Similarly, we can write

$$x^2 - 3x + 1 = (x - 1)(x - 2) - 1$$

So, on dividing $x^2 - 3x + 1$ by $x - 1$, the quotient is $x - 2$ and remainder is -1.

We can find quotient and remainder as in the case of factors. For example, let's take the quotient on dividing $x^2 - 3x - 10$ as $x - 2$ as $x - a$ and the remainder as b.

That is,

$$x^2 - 3x - 10 = (x - 2)(x - a) + b$$

Concept and meaning

We have noted that natural numbers, fractions and irrational numbers were invented to indicate different kinds of measures. And also that operation with these numbers were based on the physical context of using such measures.

When we try to divide 14 sweets among 3 children, we will have 2 sweets left over which cannot be given whole. And when we try to cut a 14 metre long string into 3 metre pieces, 2 metres are left short of length. These circumstances lead to the mathematical statement that 14 divided by 3 leaves remainder 2. If we think like this, the question, what the remainder is on dividing -14 by -3, has no meaning.

On expanding the right side of the equation, we get

$$x^2 - 3x - 10 = x^2 - (a + 2)x + (2a + b)$$

Now comparing the coefficients on each side, we get

$$a + 2 = 3$$

$$2a + b = -10$$

In this, we get $a = 1$ from the first equation and using this in the second equation we get $b = -12$.

So

$$x^2 - 3x - 10 = (x - 2)(x - 1) - 12$$

Thus on dividing $x^2 - 3x - 10$ by $x - 2$, the quotient is $x - 1$ and the remainder is -12.

Suppose we divide $x^2 - 3x - 10$ by $x + 2$ instead of $x - 2$?

Meaning of remainder

To extend the idea of remainder to all whole numbers, we must first interpert this idea in a purely mathematical way for natural numbers

The quotient q and remainder r on dividing the natural number a by the natural number b, are numbers satisfying these conditions:

1. $a = qb + r$
2. q, r are natural numbers or 0
3. $r < b$

Writing,

$$x^2 - 3x - 10 = (x + 2)(x - a) + b = x^2 - (a - 2)x + (b - 2a)$$

we get

$$a - 2 = 3$$

$$b - 2a = -10$$

from which we get $a = 5$ and $b = 0$. So

$$x^2 - 3x - 10 = (x + 2)(x - 5)$$

Thus $x + 2$ is a factor of $x^2 - 3x - 10$; we may say that the remainder is 0 in this case.

We can divide a third degree polynomial also by first degree polynomial and find quotient and remainder like this. For example, let's find the quotient and remainder on dividing $x^3 - 2x^2 - x + 4$ by $x - 3$.

Here we must note one thing before we start. It is not right to take the quotient to be the first degree polynomial $x - a$ and remainder to be the number b, as in the case of dividing a second degree polynomial. The polynomial $(x - 3)(x - a) + b$ is only of degree 2, but $x^3 - 2x^2 - x + 4$ is of degree 3.

So let's take a second degree polynomial $x^2 + ax + b$ as quotient and the number c as remainder. Then

$$x^3 - 2x^2 - x + 4 = (x - 3)(x^2 + ax + b) + c$$

Expanding the product on the right, we write

$$x^3 - 2x^2 - x + 4 = x^3 + (a - 3)x^2 + (b - 3a)x + (c - 3b)$$

From this we get,

$$a - 3 = -2$$

$$b - 3a = -1$$

$$c - 3b = 4$$

Now we get $a = 1$ from the first equation, using this in the second equation we get $b = 2$, and using this in the third we get $c = 10$.

Thus

$$x^3 - 2x^2 - x + 4 = (x - 3)(x^2 + x + 2) + 10$$

Similarly, we can find the quotient and remainder on dividing any polynomial by a first degree polynomial of the form $x - a$.

For any polynomial $p(x)$ and a polynomial of the form $x - a$, we can find a polynomial $q(x)$ and a number b such that

$$p(x) = (x - a)\, q(x) + b$$

Remainder in whole numbers

We can extend the definitions of remainder in natural numbers to whole numbers by making some changes. The quotient q and remainder r on dividing the whole number a by the whole number b are the numbers satisfying these conditions.

1. $a = qb + r$
2. q, r are whole numbers
3. $r = 0$ or $0 < r < |b|$

For example, taking –14 and –3, we have

1. $-14 = 5 \times (-3) + 1$
2. 5 and 1 are whole numbers
3. $0 < 1 < |-3|$

So we take 5 as the quotient and 1 as the remainder on dividing –14 by –3.

The equation here may be writen

$$p(x) - b = (x - a)\, q(x)$$

What does this mean?

If $p(x)$ is not a multiple of $x - a$, then by subtracting a number we can change it to a multiple.

For example, let's see what number subtracted from $x^3 - 2x^2 + x + 2$ gives a multiple of $x - 3$.

As before, we write

$$x^3 - 2x^2 + x + 2 = (x - 3)(x^2 + ax + b) + c$$

So that

$$x^3 - 2x^2 + x + 2 - c = (x - 3)(x^2 + ax + b)$$

This means, $x^3 - 2x^2 + x + 2 - c$ is a multiple of $x - 3$. So, c is the number we want. As before we can first find a, then b and finally c. There is a quicker method to find c directly. Look at the equation above again. Whatever number we take as x, the number on the two sides must be equal.

Suppose we make the right side zero?

For that we need only take $x = 3$.

$$3^3 - (2 \times 3^2) + 3 + 2 - c = (3 - 3) \times (x^2 + a \times 3 + b) = 0$$

Simplifying this, we get

$$14 - c = 0$$

$$c = 14$$

Thus we get a multiple of $x - 3$ by subtracting 14 from $x^3 - 2x^2 + x + 2$; that is, $x^3 - 2x^2 + x - 12$ is a multiple of $x - 3$.

We can use this technique to find the remainder on dividing a polynomial by a first degree polynomial of the form $x - a$. For example, let's see how we can find the remainder on dividing $x^4 + 2x^3 - 6x^2 + x + 5$ by $x - 2$.

Since we don't need the quotient, we can write it as $q(x)$. Writing the remainder as b,

$$x^4 + 2x^3 - 6x^2 + x + 5 = (x - 2)\, q(x) + b$$

We need only b. So we rewrite the equation as

$$b = (x^4 + 2x^3 - 6x^2 + x + 5) - (x - 2)\, q(x)$$

Taking $x = 2$ in this,

$$b = (2^4 + (2 \times 2^3) - (6 \times 2^2) + 2 + 5) - (2 - 2) \times q(2) = 15$$

Thus the remainder is 15.

Let's write this in general terms. To get the remainder on dividing $p(x)$ by $x - a$, we write

$$p(x) = (x - a)\, q(x) + b$$

We then rewrite the equation as

$$b = p(x) - (x - a)\, q(x)$$

Taking $x = a$ in this,

$$b = p(a) - (a - a)(q(a)) = p(a)$$

What do we see here?

The remainder on dividing the polynomial $p(x)$ by the polynomial $x - a$ is the number $p(a)$.

Let's see an example:

To find the rermainder on dividing $x^3 - 2x^2 - 4x + 5$ by $x + 2$, we first write $x + 2$ as $x - (-2)$. Now by the general result above, we need only take $x = -2$ in the first polynomial to get the remainder.

Thus the remainder is,

$$(-2)^3 - 2 \times (-2)^2 - 4 \times (-2) + 5 = -8 - 8 + 8 + 5 = -3$$

What if we divide by $2x - 1$?

We first write

$$2x - 1 = 2\left(x - \frac{1}{2}\right)$$

Next we find the remainder on dividing $x^3 - 2x^2 - 4x + 5$ by $x - \frac{1}{2}$, by taking $x = \frac{1}{2}$ in the first polynomial.

$$\left(\frac{1}{2}\right)^3 - \left(2 \times \left(\frac{1}{2}\right)^2\right) - \left(4 \times \frac{1}{2}\right) + 5 = \frac{1}{8} - \frac{1}{2} - 2 + 5 = 2\frac{5}{8}$$

Since the remainder on dividing $x^3 - 2x^2 - 4x + 5$ by $x - \frac{1}{2}$ is $2\frac{5}{8}$, we have

$$x^3 - 2x^2 - 4x + 5 = \left(x - \frac{1}{2}\right) q(x) + 2\frac{5}{8}$$

This can be slightly changed to this:

$$x^3 - 2x^2 - 4x + 5 = \frac{1}{2}(2x - 1)\, q(x) + 2\frac{5}{8}$$

Now writing $\frac{1}{2} q(x)$ as $r(x)$, the equation becomes

$$x^3 - 2x^2 - 4x + 5 = (2x - 1)\, r(x) + 2\frac{5}{8}$$

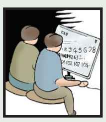
Thus the remainder on dividing $x^3 - 2x^2 - 4x + 5$ by $2x - 1$ is also $2\frac{5}{8}$.

We can also use this general principle to check whether a first degree polynomial is a factor of another polynomial. If the remainder is zero, then we have a factor, right? By the general result, the remainder on dividing $p(x)$ by $x - a$ is $p(a)$. So if $p(a) = 0$, then $x - a$ is a factor of $p(x)$.

For a polynomial $p(x)$ and a number a, if $p(a) = 0$, then $x - a$ is a factor of $p(x)$.

We saw earlier that if $x - a$ is a factor of $p(x)$, then $p(a) = 0$. Now we can have the converse also

How do we find the remainder on dividing a polynomial $p(x)$ by a polynomial of the form $ax + b$ is a factor? How do we check whether $ax + b$ is a factor?



(1) For each pair of polynomials below, check whether the first is a factor of the second. If not a factor, find the remainder on dividing the second by the first.

i) $x - 1,\ x^3 + 4x^2 - 3x - 6$ ii) $x + 1,\ x^3 + 4x^2 - 3x - 6$

iii) $x - 2,\ x^3 + 3x^2 - 4x - 12$ iv) $x + 2,\ x^3 + 3x^2 - 4x - 12$

v) $2x - 1,\ 2x^3 - x^2 - 8x + 6$ vi) $3x - 1,\ 3x^3 - 10x^2 + 9x - 2$

(2) For each pair of polynomials below, find the quotient and remainder on dividing the first by the second.

i) $x^3 - 1,\ x - 1$ ii) $x^3 - 1,\ x + 1$

iii) $x^3 + 1,\ x - 1$ iv) $x^3 + 1,\ x + 1$

(3) By adding a number to $p(x) = x^3 + x^2 + x$, a new polynomial $q(x)$ is to be formed.

i) What number should be added, so that $x - 1$ is a factor of $q(x)$?

ii) What number should be added, so that $x + 1$ is a factor of $q(x)$?

(4) In each pair of polynomials below find what kind of natural number n must be, so that the first is a factor of the second.

i) $x - 1, x^n - 1$ ii) $x - 1, x^n + 1$ iii) $x + 1, x^n - 1$

iv) $x + 1, x^n + 1$ v) $x^2 - 1, x^n - 1$

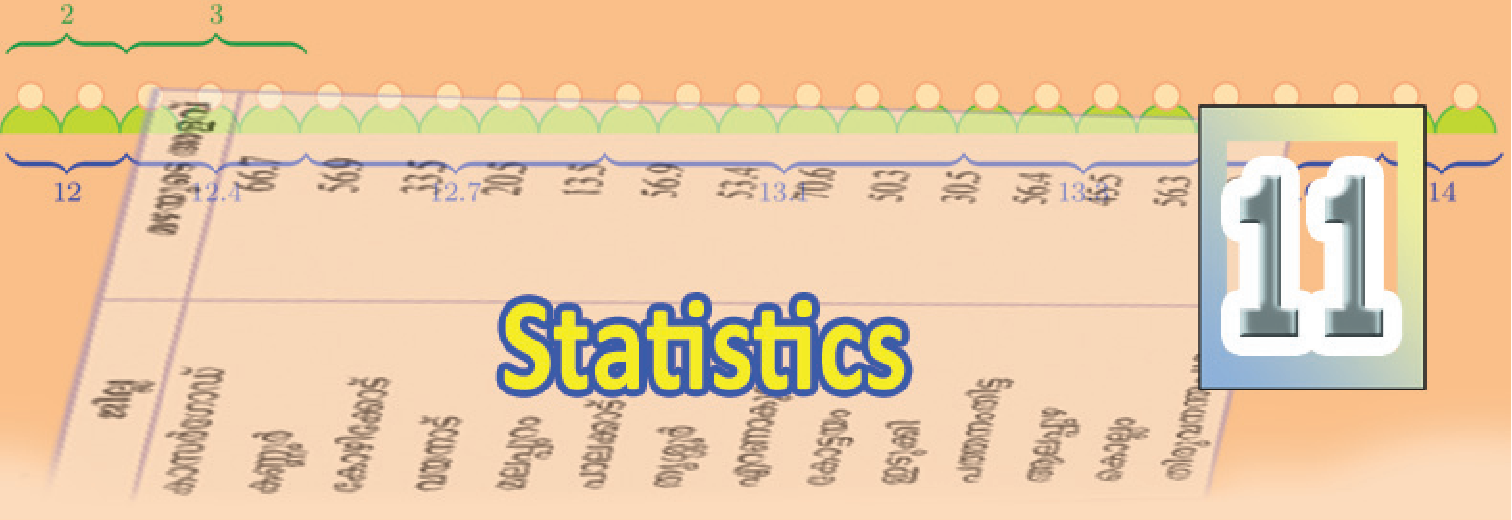
(5) Prove that if $x^2 - 1$ is a factor of $ax^3 + bx^2 + cx + d$, then $a = -c$ and $b = -d$.

(6) What first degree polynomial added to $2x^3 - 3x^2 + 5x + 1$ gives a multiple of $x^2 - 1$?

What is the relation between the coefficients of a polynomial which is a multiple of $x^2 - 4$? What if we take $x^2 - 9$ instead of $x^2 - 4$?

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Explaning methods to factorize a second degree polynomial as a product of first degree polynomials. • Explaning the method of check if $x - a, x + a$ are factors of $p(x)$. • Calculating the remainder on dividing a polynomial by a first degree polynomial, without actual division.			



11

Statistics

Not a correct average

The monthly incomes of 10 households in a neighbourhood are these:

16500	21700	18600	21050	19500
17000	21000	18000	22000	17500

What is the mean monthly income?

Adding all these and dividing by 10, we get the mean monthly income as 19285 rupees.

Now if instead of taking all these incomes separately, we had only the mean, then also we can make some conclusions about the general economic status of the households:

- The monthly incomes of all these households are around 19285 rupees.
- None of the households has a monthly income very much greater or very much less than 19285 rupees.
- The number of households with monthly income greater than 19285 rupees is more or less equal to the number of households with monthly income less than 19285 rupees.

Now suppose some one with a monthly income of 175000 rupees comes to live in the neighborhood. What is the mean monthly income of the 11 households?

$$\frac{(19285 \times 10) + 175000}{11} \approx 33441 \text{ rupees.}$$

Without giving all these details, if this mean only is given, wouldn't we make the wrong conclusion that all these households have a monthly income around 30000 rupees? This is almost one and a half times the monthly income of ten of these households.

The purpose of calculating the mean is to reduce a whole collection of numbers to a single number, which gives a general understanding of a situation. But numbers in the collection which is very much less or very much more than others (though few) affect the mean a lot.

In our example, it was a single number very much larger than the first ten which changed the mean so much. Can you think of other instances like this where very small or very large numbers influence the mean to give a wrong impression?

Another average

Let's see how we can compute another average which gives a better overall indication of the monthly income of the 11 households. If we write all the incomes in increasing order and take the middle number, 5 of the households would have income less than this and 5 of them would have more.

Let's write the numbers in order:

16500, 17000, 17500, 18000, 18600, 19500, 21000,
21050, 21700, 22000, 175000

The middle number is 19500. It is called the *median* of these numbers. That is, the median monthly income of the 11 households is 19500 rupees. We can put it like this: of all the 11 house heads, 5 have monthly income less than 195000 rupees and 5 have more than 19500 rupees. That is, the number of households with income less than the median and the number of households with income more than the median are equal.

What if we take only the first 10 households? If we write incomes in the increasing order, there would be two numbers, 18600 and 19500 at the middle, instead of just one number.

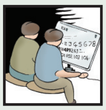
Here also, we must choose the median such that the number of items below it and above it are equal. Any number between 18600 and 19500 would do for this. Usually half their sum is taken as the median.

That is, the median monthly income of the first 10 households is

$\frac{1}{2}$ (18600 + 19500) = 19050 rupees

The median income 19500 rupees, like mean income 19285 rupees gives a reasonable estimate of the economic status of the first ten households (and there is no great difference between the mean and the median either).

What is important here is that the high income of the eleventh household does not change the median much. Also if we say that the median income of some households is 19050 rupees and that the monthly income of one of these is 21000 rupees, we can conclude that this household is better off than more than half the households considered.



(1) The distance covered by an athlete in long jump practice are

6.10, 6.20, 6.18, 6.20, 6.25, 6.21, 6.15, 6.10

in metres. Find the mean and median. Why is it that there is not much difference between these?

(2) The table below gives the rainfall during one week of september 2015 in various districts of Kerala.

District	Rainfall (mm)
Kasaragod	66.7
Kannur	56.9
Kozhikode	33.5
Wayanad	20.5
Malappuram	13.5
Palakkad	56.9
Thrissur	53.4
Ernakulam	70.6
Kottayam	50.3
Idukki	30.5
Pathanamthitta	56.4
Alapuzha	45.5
Kollam	56.3
Thiruvananthapuram	89.0

Calculate the mean and median rainfall in Kerala during this week. Why is the median less than mean?

(3) Prove that for a set of numbers in arithmetic sequence, the mean and median are equal.

Frequency and median

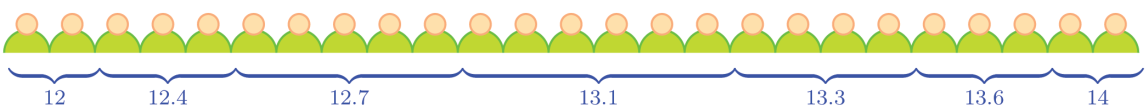
The amount of Hemoglobin in blood is usually given as grams per decilitre (that is, 100 millilitres). The table below shows 25 children sorted according to hemogoblin levels, after a blood test.

Hemoglobin (g/dl)	Number of children
12.0	2
12.4	3
12.7	5
13.1	6
13.3	4
13.6	3
14.0	2

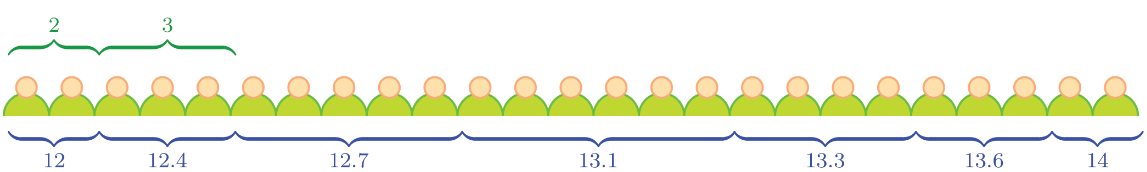
From this, we can compute the mean hemoglobin level. How do we find the median?

Median is that which comes in the middle; that is 12 of the children should have less than the median level and 12 more than the median level.

To find it, we need only make the kids stand in a line, in the order of hemoglobin level and take the level of the thirteenth kid. Imagine the kids standing like this;

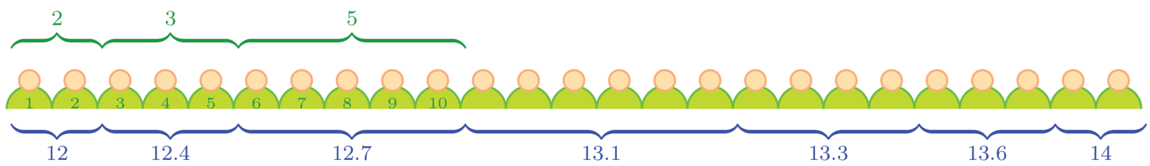


The first two have hemogloblin 12, the next 3 have 12.4 and thus the line grows.



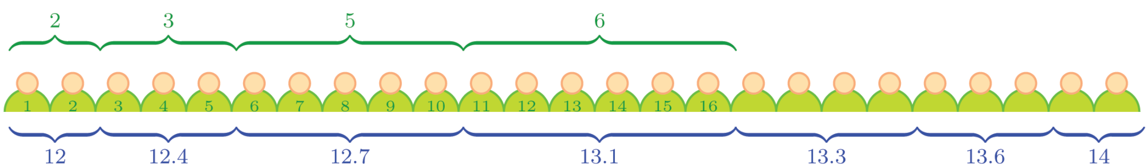
We want the hemoglobin level of the 13^{th} kid. By adding the numbers in the table one by one, we can find his position in the hemoglobin sequence. Taking the $2 + 3 = 5$ kids of the first two groups, the level rises to 12.4. That is the 5^{th} kid has level 12.4.

Adding the 3 kids in the next group, we have 5 + 5 = 10 kids and the level reaches 12.7.

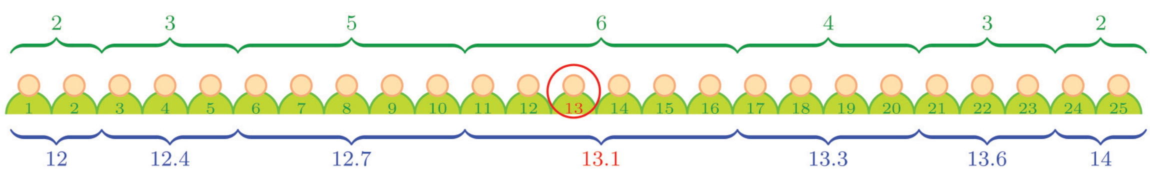


Thus the 10th kid is at level 12.7.

Adding the 6 kids in the next group, we have 10 + 6 = 16 kids.



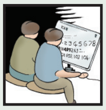
We need the level of the 13th kid. The level of all the kids from the 11th to the 16th in the line is 13.1. So the 13th kid also has this level, and this is the median level.



We can do this using a table instead of a picture:

Hemoglobin (g/dL)	Number of children
up to 12.0	2
up to 12.4	5
up to 12.7	10
up to 13.1	16
up to 13.3	20
up to 13.6	23
up to 14.0	25

From the table, we can see the hemoglobin levels of the kids, from the 11th to the 16th, is 13.1. Since the middle one of 25, that is the 13th, is in this set, the median level can be found as 13.1.



(1) 15 households in a neighbourhood are sorted according to their monthly income in the table below.

Monthly income (Rs)	Number of households
4000	3
5000	7
6000	8
7000	5
8000	5
9000	4
10000	3

Calculate the median income.

(2) The table below shows the workers in a factory sorted according to their daily wages:

Daily wages (Rs)	Number of workers
400	2
500	4
600	5
700	7
800	5
900	4
1000	3

Calculate the median daily wage.

(3) The table below gives the number of babies born in a hospital during a week, sorted according to their birth weight.

Weight (kg)	Number of babies
2.500	4
2.600	6
2.750	8
2.800	10
3.000	12
3.150	10
3.250	8
3.300	7
3.500	5

Calculate the median birth-weight.

Classes and median

The table below shows the children in a class, sorted according to their heights.

Height (cm)	Number of children
135 – 140	5
140 – 145	8
145 – 150	12
150 –- 155	11
155 – 160	5
160 – 165	4
Total	**45**

How do we compute the median height of children of this class?

We have to find the height of the kid in the middle, when they are made to stand in a line, in the order of their heights. There are 45 kids and so the one in the middle is the 23rd kid.

In the table, height is divided into various classes. Let's first see which class the 23rd kid comes in. As before, let's count the total number of kids as each class is added.

Height	Number
Below 140	5
Below 145	13
Below 150	25
Below 155	36
Below 160	41
Below 165	45

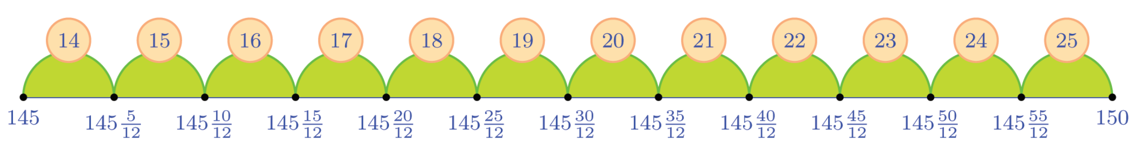
According to this table, when we take kids upto 145 centimetres tall, we reach upto the 13th kid, adding those upto 150 centimetres tall, we reach upto the 25th kid. The 23rd kid we want is between these two. Thus we see that his height is between 145 and 150 centimetres.

How do we make it more exact?

We only know that the 12 kids from the 14th to the 25th are of heights between 145 and 150 centimetres, we don't know their individual heights.

So we have to make some assumptions. (Recall making some assumptions in computing the mean from a frequency table)

We divide the 5 centimetres from 145 to 150 centimetres into 12 equal parts and assume that one kid is in each such subdivision.

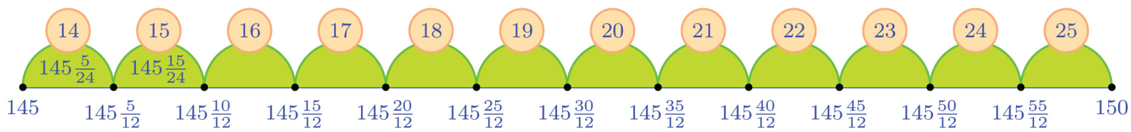


We also assume that the height of each kid in a subdivision is the mid value of this subdivision. So the height of the 14th kid is the mid value of 145 and $145\frac{5}{12}$ centimetres.

That is, $145\frac{5}{24}$ centimetres.

What about the next kid?

His height is the mid value of $145\frac{5}{12}$ and $145\frac{10}{12}$; that is, $145\frac{15}{24}$ centimetres.



Can you compute the height of each? What is the rate of change of height?

Let's look at our computations so far:

- The height of the 14th kid is $145\frac{5}{24}$ centimetres.
- The height of each kid thereafter increases by $\frac{5}{12}$ centimetres.
- From the 14th to 23rd, there are 9 kids.

So the problem now is that of an arithmetic sequence:

If the 14th term of an arithmetic sequence is $145\frac{5}{24}$ and the common difference is $\frac{5}{12}$, what is the 23rd term?

Thus the height of the 23rd kid is

$$145\frac{5}{24} + \left(9\times\frac{5}{12}\right) = 145 + \frac{95}{24} = 148\frac{23}{24} \approx 148.9 \text{ centimetres.}$$

Let's now do a problem like this, without drawing any picture.

The table below gives the number of persons working in an office, sorted according to their ages.

Age	Number of workers
25 – 30	6
30 – 35	8
35 – 40	12
40 – 45	20
45 – 50	16
50 – 55	6
Total	**68**

We have to find the median age. Since the number of workers is the even number 68, we arrange them in the order of age, take the persons in the 34^{th} and 35^{th} positions, and then find half the sum of their ages. First we write the cumulative frequencies:

Age	Number of workers
Below 30	6
Below 35	14
Below 40	26
Below 45	46
Below 50	62
Below 55	68

According to this, the ages of the 20 workers from the 27^{th} to the 46^{th} position, in the order of age, is between 40 and 45 years. The workers in the 34^{th} and 35^{th} positions we need are in this group.

As before, we divide the 5 years from 40 to 45 into 20 equal parts and assume that each subdivision contains one worker whose age is the mid value of the subdivision. So the length of each subdivision is $\frac{5}{20} = \frac{1}{4}$ of a year.

So, the age of the 27^{th} worker is the mid value of 40 and $40\frac{1}{4}$; that is $40\frac{1}{8}$ years. Our assumtion is that the age of each one thereafter increases by $\frac{1}{4}$ year. So the age of the 34^{th} worker is,

$$40\frac{1}{8} + \left(7\times\frac{1}{4}\right) = 40 + \frac{15}{8} = 41\frac{7}{8} \text{ years.}$$

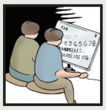
Age of the 35^{th} person is

$$41\frac{7}{8} + \frac{1}{4} = 42\frac{1}{8} \text{ years.}$$

Now to find the median age, we calculate half the sum of these ages.

$$\frac{1}{2}\left(41\frac{7}{8} + 42\frac{1}{8}\right) = \frac{1}{2} \times 84 = 42$$

Thus the median age is 42 years.



(1) Some households in a locality are sorted according to their electricity usage and given in the table below:

Usage of electricity (units)	Number of households
80 – 90	3
90 – 100	6
100 – 110	5
110 – 120	8
120 – 130	9
130 – 140	4

Calculate the median usage.

(2) The table below shows children in a class sorted according to their marks in a maths exam:

Marks	Number of children
0 – 10	4
10 – 20	10
20 – 30	12
30 – 40	9
40 – 50	5

Calculate the median mark.

(3) The income tax paid by the workers in an office is shown below:

Income tax (Rs)	Number of workers
1000 – 2000	8
2000 – 3000	10
3000 – 4000	15
4000 – 5000	18
5000 – 6000	22
6000 – 7000	8
7000 – 8000	6
8000 – 9000	3

Calculate the median tax paid.

Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Recognising the contexts where the mean cannot be used to represent a set of measures. • Explaining the method to compute the median of a set of measures. • Explaining the method to compute the median from a set of measures given as a frequency table.			

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